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# PyRsw Documentation

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Contents:



List of problems to solve

1. solve 1d 1-layer SW (periodic/boundaries)
2. solve 1d n-layer SW
3. solve 2d 1-layer SW
4. solve 2d n-layer SW
5. Linear Stability Calculations

Equations and latex to include:

1. derivation of SW
2. governing equations for different models (see above)
3. different numerical methods:
  - (a) FD Sardouney
  - (b) spectral
  - (c) WENO
  - (d) f2py
  - (e) openmp
  - (f) openmp + mpi





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## One-Layer Rotating Shallow Water Model

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### 2.1 Classical Form

The one-layer, two-dimensional rotating shallow water model on a rotating  $f$ -plane can be written as,

$$\begin{aligned}\frac{\partial u}{\partial t} + (\mathbf{u} \cdot \nabla) u - f v &= -g \frac{\partial h}{\partial x}, \\ \frac{\partial v}{\partial t} + (\mathbf{u} \cdot \nabla) v + f u &= -g \frac{\partial h}{\partial y}, \\ \frac{\partial h}{\partial t} + \nabla \cdot (h \mathbf{u}) &= 0.\end{aligned}$$

This describes the motion of a pancake like fluid in that it is thin in the vertical and much longer in the horizontal. It contains pressure forces due to the free surface and a Coriolis pseudo-force because of the rotating frame of reference.

The fluid moves as columns that can be translated in the horizontal and stretched/contracted in the vertical. If the height of a column changes then the vorticity must change, as can be reflected in the fact that, in the absence of forcing and dissipation, Potential Vorticity is conserved following the motion,

$$\frac{D}{Dt} \left( \frac{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f}{h} \right) = 0.$$

### 2.2 Conservation Form

There are a variety of forms, these equations can be written. One of them is conservation where the three fields that have time derivatives are  $U = hu$ ,  $V = hv$  and  $h$ .

$$\begin{aligned}\frac{\partial U}{\partial t} + \frac{\partial}{\partial x} \left( \frac{U^2}{h} + \frac{gh^2}{2} \right) + \frac{\partial}{\partial y} \left( \frac{UV}{h} \right) - fV &= 0, \\ \frac{\partial V}{\partial t} + \frac{\partial}{\partial x} \left( \frac{UV}{h} \right) + \frac{\partial}{\partial y} \left( \frac{V^2}{h} + \frac{gh^2}{2} \right) + fU &= 0, \\ \frac{\partial h}{\partial t} + \nabla \cdot (h \mathbf{u}) &= 0.\end{aligned}$$

Conservation form is attractive because it can help to ensure that, using a clever numerical scheme, some quantities are conserved. Note that if you have topography then there are source terms that appear on the right-hand side and this causes some problems.

## 2.3 Vorticity-Bernoulli Form

A third form arises from rewriting the nonlinear acceleration terms as a gradient and a cross product term. Using a vector identity, it can be shown that the above system is mathematically equivalent to the following,

$$\begin{aligned}\frac{\partial u}{\partial t} - qhv &= -\frac{\partial B}{\partial x}, \\ \frac{\partial v}{\partial t} + qhu &= -\frac{\partial B}{\partial y}, \\ \frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) &= 0.\end{aligned}$$

Note that above we have defined the vorticity and the Bernoulli function,

$$\begin{aligned}q &= \frac{\zeta + f}{h} = \frac{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f}{h}, \\ B &= gh + \frac{1}{2}(u^2 + v^2).\end{aligned}$$

Before we look at solving this complicated set of equations we consider the one and a half dimensional limit.

## 2.4 $1\frac{1}{2}$ Dimensional Limit

We assume that none of the variables depend on one horizontal direction, say  $y$ . But, it is very important to realize that the velocity in that direction is not necessarily zero. Indeed, if you have flow in the  $x$ -direction, the Coriolis force will deflect it to the right, which will then generate a flow that is perpendicular. This will continue and often give rise to inertial oscillations in the horizontal. So we can have motion in either direction but the motion only changes with respect to  $x$ .

If we simplify the governing equations we get

$$\begin{aligned}\frac{\partial u}{\partial t} - qhv &= -\frac{\partial B}{\partial x}, \\ \frac{\partial v}{\partial t} + qhu &= 0, \\ \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) &= 0.\end{aligned}$$

where the vorticity and the Bernoulli function simplify to,

$$\begin{aligned}q &= \frac{\zeta + f}{h} = \frac{\frac{\partial v}{\partial x} + f}{h}, \\ B &= gh^2 + \frac{1}{2}(u^2 + v^2).\end{aligned}$$

## 2.5 Conserved Quantities

In the purely conservative or nondissipative limit, there are three quantities that are exactly conserved.

1. Mass:

$$M = \int_D h dA$$

2. Total Energy: sum of potential and kinetic energies

$$E = \frac{1}{2} \int_D (gh^2 + h(u^2 + v^2)) dA$$

3. Potential Enstrophy:

$$Q = \frac{1}{2} \int_D hq^2 dA$$

In a numerical model we cannot expect these to be conserved but we would like them to be close to be conserved. If they are very badly conserved than this could reflect that the numerical scheme is behaving badly. However, just because these are conserved that does not guarantee that the solution is correct. But they are usually good indicators as to how our method is doing in the conservative limit.

Of course when nonconservative forces are introduced things will change. One might argue that since the world is non-dissipative then we don't need to worry about conserving these. However, it is desirable to know that basis of your model is well behaved and therefore why we should worry about conserved quantities.



## Examples

### 3.1 Geostrophic Adjustment: One-Dimension

In the directory `examples` you will find an example entitled `example_1D_geoadjust.py`

First, libraries are imported. Two standard ones are `numpy`, for calculations, and `matplotlib.pyplot` for plotting. Those are standard to `numpy`. Then, there are four other things that are imported:

- **Steppers** This contains different time-stepping functions. At the moment we have Euler, Adams-Bashforth 2 (AB2), Adams-Bashforth 3 (AB3) and Runge-Kutta 4 (RK4). PyRsw uses adaptive time stepping to try and be more efficient in how the solution is marched forward.
- **Fluxes** This contains the fluxes for the RSW model. At the moment there is only the option for a pseudo-spectral model but this will be generalized to include a Finite Volume method as well.
- **PyRsw** This is the main library and importing `Simulation` imports the core of the library.
- **constants** This has some useful constants, more can be added if desired.

After the libraries are imported then a simulation object is created.

```
sim = Simulation()
```

Below specifies the geometry in  $x$  and  $y$ : [Options: 'periodic', 'walls']

We use AB3, a spectral method: [Options: Euler, AB2, AB3, RK4]

We solve the nonlinear dynamics: [Options: Linear and Nonlinear]

Use spectral sw model (no other choices at present).

```
sim.geomy      = 'periodic'
sim.stepper    = Step.AB3
sim.method     = 'Spectral'
sim.dynamics   = 'Nonlinear'
sim.flux_method = Flux.spectral_sw
```

We specify a lot of parameters. There are some default values that are specified in PyRsw.

```
sim.Ly = 4000e3      # Domain extent          (m)
sim.Nx = 1           # Grid points in x
sim.Ny = 128        # Grid points in y
sim.Nz = 1           # Number of layers
sim.g   = 9.81       # Gravity          (m/sec^2)
sim.f0  = 1.e-4      # Coriolis         (1/sec)
sim.beta = 0e-10     # Coriolis beta parameter (1/m/sec)
```

```
sim.cfl = 0.05           # CFL coefficient           (m)
sim.Hs  = [100.]         # Vector of mean layer depths (m)
sim.rho = [1025.]        # Vector of layer densities  (kg/m^3)
sim.end_time = 2*24.*hour # End Time           (sec)
```

There is an option to thread the FFTW's if using pyfftw.

```
sim.num_threads = 4
```

Below we specify the plotting interval, what kind of plotting to do, and the limits on the three figures.

```
sim.plott = 20.*minute # Period of plots
sim.animate = 'Anim'   # 'Save' to create video frames,
                       # 'Anim' to animate,
                       # 'None' otherwise
sim.ylims=[[-0.18,0.18],[-0.18,0.18],[-0.5,1.0]]
```

We can specify the periodicity of plotting and whether we want a life animation or make a video. More on this this later.

```
sim.output = False     # True or False
sim.savet  = 1.*hour   # Time between saves
```

Specify periodicity of diagnostics and whether to compute them. This is not tested.

```
sim.diagt  = 2.*minute # Time for output
sim.diagnose = False   # True or False
```

Initialize the simulation.

```
sim.initialize()
```

Specify the initial conditions. There is an option whether we want the domain in  $x$  or  $y$ . At the moment there is no difference because there is no  $\beta$ -plane but this will be added.

```
for ii in range(sim.Nz): # Set mean depths
    sim.soln.h[:, :, ii] = sim.Hs[ii]

# Gaussian initial conditions
x0 = 1.*sim.Lx/2.        # Centre
W  = 200.e3              # Width
amp = 1.                 # Amplitude
sim.soln.h[:, :, 0] += amp*np.exp(-(sim.Y)**2/(W**2))
```

Solve the problem.

```
sim.run()
```

Plot the Hovmöller diagram in time versus space.

```
# Hovmuller plot
plt.figure()
t = np.arange(0, sim.end_time+sim.plott, sim.plott)/86400.

if sim.Ny==1:
    x = sim.x/1e3
elif sim.Nx == 1:
    x = sim.y/1e3

for L in range(sim.Nz):
    field = sim.hov_h[:, 0, :].T - np.sum(sim.Hs[L:])
```

```

cv = np.max(np.abs(field.ravel()))
plt.subplot(sim.Nz,1,L+1)
plt.pcolormesh(x,t, field,
               cmap=sim.cmap, vmin = -cv, vmax = cv)
plt.axis('tight')
plt.title(r"$\mathrm{Hovm}(\mathrm{o}ller) \ ; \ \mathrm{Plot} \ ; \ \mathrm{of} \ ; \ \eta$", fontsize = 16)
if sim.Nx > 1:
    plt.xlabel(r"$\mathrm{x} \ ; \ \mathrm{(km)}$", fontsize=14)
else:
    plt.xlabel(r"$\mathrm{y} \ ; \ \mathrm{(km)}$", fontsize=14)
plt.ylabel(r"$\mathrm{Time} \ ; \ \mathrm{(days)}$", fontsize=14)
plt.colorbar()

plt.show()

```

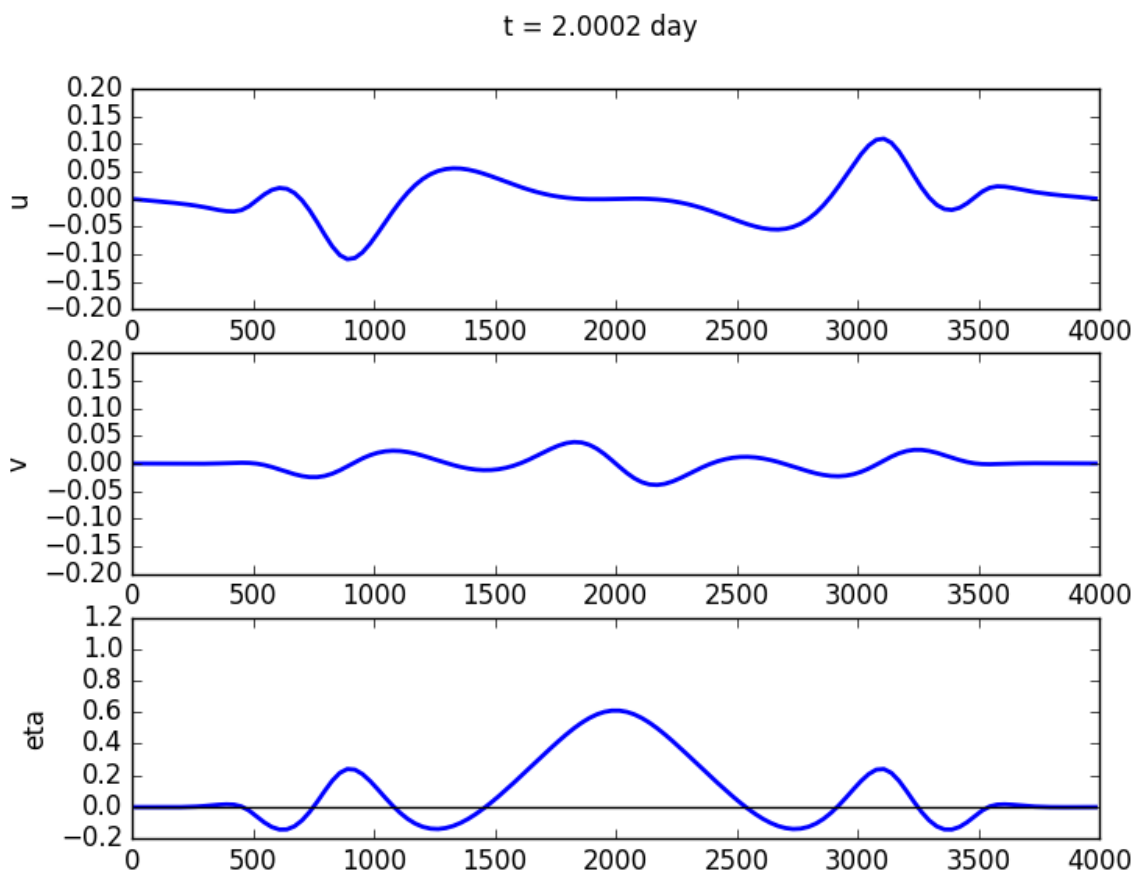


Fig. 3.1: Final solution for the test case.

<<<<<<< HEAD There is a second example called `example_1D_geoadjust2.py` that begins with a hyperbolic tangent profile instead of a Gaussian initial condition. ===== Note that to compute the derivatives in the case of a non-periodic domain we impose either Dirichlet or Neumann boundary conditions. This is done by doing odd and even extensions respectively. That is why in 1D, the simulation with walls does twice as much work as in the periodic case. Similarly, if we have walls in 2D, that is doing four times as much work.

At some point we should change walls to 'slip' and allow for 'noslip' boundary conditions as well.

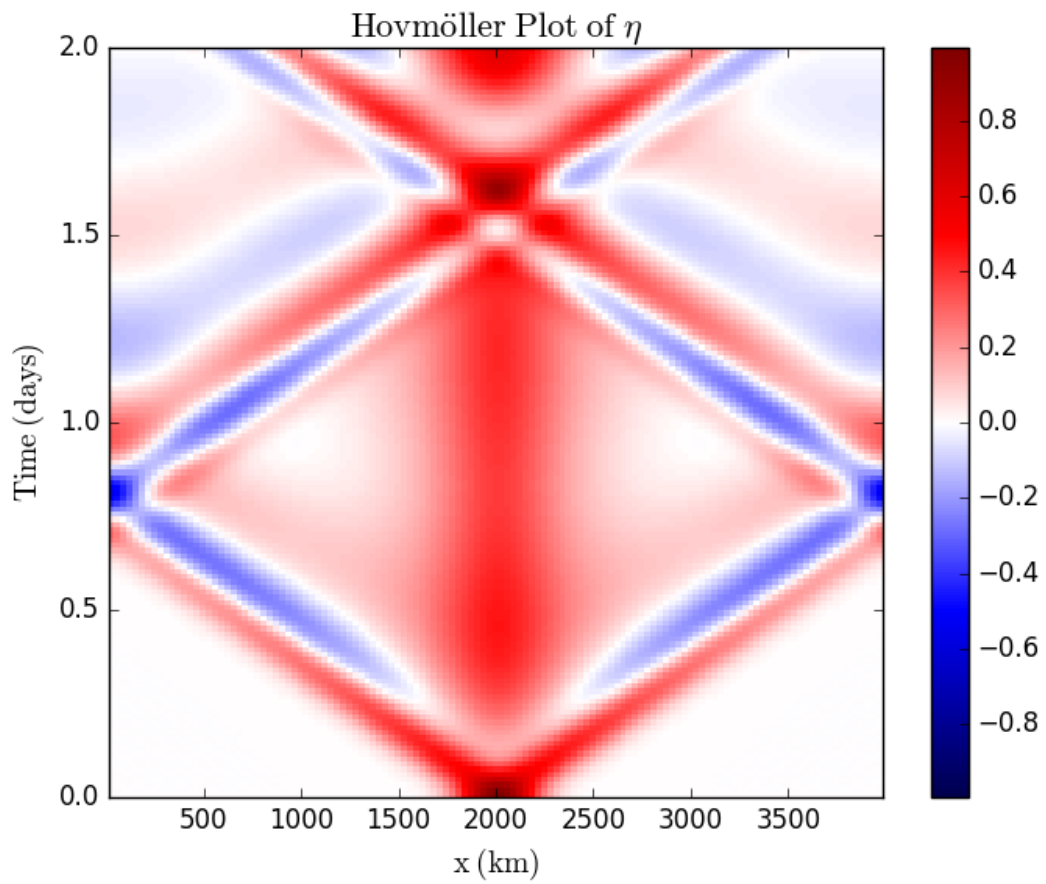


Fig. 3.2: Hovmöller plot for the test case.



## 3.2 Geostrophic Adjustment: Two-Dimensions

The basic script is almost identical to the 1D case and can be found in the examples folder with the title `example_2D_geoadjust.py`. The changes are as follows:

- Set  $N_x$  and  $N_y$  both equal to 128, and from this we build a 2D grid.
- Specify the length of the domain in the zonal direction.
- Define the initial conditions on a 2D grid.
- The plotting is different. We plot a 2D field using `pcolormesh` and we don't do a Hovmöller plot.

## 3.3 Bickley Jet: Two-Dimensions

Following Poulin and Flierl (2003) and Irwin and Poulin (2014), we look at the instability of a Bickley jet. The script is called `example_2D_BickleyJet.py`.

In this case we change the code to include the following lines.

```
# Define geometry
sim.geomx      = 'periodic'
sim.geomy      = 'walls'

# Define grid and domain size
sim.Lx = 200e3      # Domain extent          (m)
sim.Ly = 200e3      # Domain extent          (m)
sim.Nx = 128        # Grid points in x
sim.Ny = 128        # Grid points in y

# Bickley Jet initial conditions
# First we define the jet
Ljet = 20e3         # Jet width
amp = 0.1          # Elevation of free-surface in basic state
sim.soln.h[:, :, 0] += -amp*np.tanh(sim.Y/Ljet)
sim.soln.u[:, :, 0] = sim.g*amp/(sim.f0*Ljet)/(np.cosh(sim.Y/Ljet)**2)
# Then we add on a random perturbation
sim.soln.u[:, :, 0] += 2e-3*np.exp(-(sim.Y/Ljet)**2)*np.random.randn(sim.Nx, sim.Ny)
```

```
>>>>>> master
```

## 3.4 Geostrophic Adjustment: 2D and 1L

The basic script is almost identical to the 1D case. The changes are as follows:

- Set  $N_x$  and  $N_y$  both equal to 128, and from this we build a 2D grid.
- Define the initial conditions on a 2D grid.
- The plotting is different. We plot a 2D field using and we don't do a Hovmöller plot.

## 3.5 Bickley Jet: 2D and 1L

Following Poulin and Flierl (2003) and Irwin and Poulin (2014), we look at the instability of a jet.



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## Linear Stability Analysis

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### 4.1 One-Layer Shallow Water

In this subsection we consider the one-layer reduced gravity RSW model with topography below. We define the following:

- $H$ : mean depth of the layer
- $z = \eta$ : height of the free surface
- $z = -H + \eta_B$ : height of the topography.
- $h = H + \eta - \eta_B$ : total depth of layer
- $(u, v)$ : horizontal velocity
- $g'$ : reduced gravity
- $\rho_0$ : reference density

The governing nonlinear equations are,

$$\begin{aligned}\frac{\partial u}{\partial t} + \vec{u} \cdot \vec{\nabla} u - fv &= -g \frac{\partial}{\partial x} (h + \eta_B), \\ \frac{\partial v}{\partial t} + \vec{u} \cdot \vec{\nabla} v + fu &= -g \frac{\partial}{\partial y} (h + \eta_B), \\ \frac{\partial h}{\partial t} + \vec{\nabla} \cdot (h\vec{u}_1) &= 0.\end{aligned}$$

### 4.2 Basic State

To study shear flows in a meridional channel we consider solutions of the form,

$$\begin{aligned}u &= U_B(y), \\ v &= 0, \\ h &= H_B(y).\end{aligned}$$

For this to be an exact solution we require that the flow is in geostrophic balance,

$$fU_B = -g \frac{d}{dy} (H_B + \eta_B).$$

### 4.3 Perturbation

We perturb the basic state with infinitesimal quantities,

$$\begin{aligned} u &= U_B(y) + u', \\ v &= 0 + v', \\ h &= H_B(y) + h'. \end{aligned}$$

We substitute our perturbation into the governing equations and drop the primes (for brevity) and cancelling out the geostrophic terms

$$\begin{aligned} \frac{\partial u}{\partial t} + (u + U_B) \frac{\partial u}{\partial x} + v \frac{\partial}{\partial y} (u + U_B) - fv &= -g \frac{\partial h}{\partial x}, \\ \frac{\partial v}{\partial t} + (u + U_B) \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + fu &= -g \frac{\partial h}{\partial y}, \\ \frac{\partial h}{\partial t} + (u + U_B) \frac{\partial h}{\partial x} + v \frac{\partial}{\partial y} (H_B + h) + (H_B + h) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0. \end{aligned}$$

Now we neglect the quadratic terms to obtain the linearized equations,

$$\begin{aligned} \frac{\partial u}{\partial t} &= -U_B \frac{\partial u}{\partial x} + \left( f - \frac{dU_B}{dy} \right) v - g \frac{\partial h_1}{\partial x}, \\ \frac{\partial v}{\partial t} &= -fu - U_B \frac{\partial v}{\partial x} - g \frac{\partial h_1}{\partial y}, \\ \frac{\partial h}{\partial t} &= -H_B \frac{\partial u}{\partial x} - v \frac{dH_B}{dy} - H_B \frac{\partial v}{\partial y} - U_B \frac{\partial h}{\partial x}. \end{aligned}$$

Finally, we assume a normal mode decomposition in the zonal direction and time,

$$= \text{Re} \left\{ e^{ik(x-ct)} [\hat{u}, ik\hat{v}, \hat{h}] \right\},$$

which we can substitute into the above equations to yield in the inviscid limit

$$\begin{aligned} c\hat{u} &= U_B \hat{u} - \left( f - \frac{dU_B}{dy} \right) \hat{v} + g\hat{h}, \\ c\hat{v} &= -\frac{f}{k^2} \hat{u} + U_B \hat{v} - \frac{g}{k^2} \frac{d\hat{h}}{dy}, \\ c\hat{h} &= H_B \hat{u} + \frac{d}{dy} (H_B \hat{v}) + U_B \hat{h}. \end{aligned}$$

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## Indices and tables

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- `genindex`
- `modindex`
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