Poincaré Project Documentation

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This is an experiment to put my blog posts from

http://jtauber.com/poincare_project/

into a git repository to track history, allow people to file issues, fork and make pull requests.

The repository is available at

https://github.com/journeymanofsome/poincare_project

I'm porting over one post at a time into Sphinx and publishing them to

http://poincare-project.readthedocs.org/

Here is how I described the original content on my blog:

> I am currently working through the mathematics required to understand the Poincaré Conjecture and the possible solution recently proposed. I want to blog my journey and I started out summarising the basic foundations of pure mathematics necessary to get to the conjecture-specific parts. Now that I've got to explaining the conjecture, posts are about the background in differential geometry to understand the proofs.

This is a flat list of the posts converted so far. Lots more to come then I'll look and structuring them a little more.

CHAPTER 1

Thinking Like a Pure Mathematician

Before we are at a point where we can discuss the Poincare Conjecture itself, we need to learn some general topology and group theory. But before we lay that foundation, I think it is worth taking a moment to establish the mode of thinking we must enter.

Marcus Aurelius exhorts us to ask “what is the nature of the whole, and what is my nature, and how this is related to that, and what kind of a part it is of what kind of a whole?” Now Aurelius is talking about human nature (and see Hannibal Lecter’s use of the quotation in Silence of the Lambs) but it encapsulates the fundamental questions asked by pure mathematicians, not of humans, but of abstract objects such as numbers and shapes.

Imagine you’re looking at an apple and you notice certain characteristics it posseses. Which of those characteristics are specific to that particular apple? Which are specific to all apples of that particular variety? Of apples in general? Or of all fruit? Of food? Organic objects? Physical objects?

In mathematics in general, and in the early days of this Poincare Project in particular, we will often be asking questions like: what is the most general object that exhibits this characteristic? What is the distinguishing characteristic of this object compared with others we’re dealing with?

Get your mind in a mode to ask those kind of questions and we’ll be ready to introduce topology.

CHAPTER 2

Adding Structure to Sets

Most pure mathematics takes as a starting point a set of objects. If things stopped there, we would be dealing just with set theory; but we branch into other areas of pure mathematics by adding structure to the set.

Defining such a structure may involve calling out particular elements of the set or particular subsets whose element have some particular relationship to one another; or it may involve some mapping from one element in the set to other or an operation that takes two elements of the set and produces a third.

Calling out particular subsets is the basis for the branch known as *topology*. If the choice of subsets meets certain criteria (which we'll get to shortly), the set (along with the called-out subsets) is called a *topological space*.

Defining an operation that takes two elements of the set and produces a third that is also in the set (think of adding two numbers or concatenating two strings) is the basis for the branch known as *group theory*. If the operation meets certain criteria (which we'll also get to shortly), the set and the operation is called a *group*.

Some structures involve reference to one or more additional sets (such as the set of real numbers). For example, one might define an operation that takes two elements of a set and gives a number that can be thought of as the "distance" between the two elements. As long as that operation follows certain rules (such as the distance between two distinct elements always being positive and the distance between an element and itself always being 0) then the operation is called a *metric* and the set and the operation is called a *metric space*.

In the next few entries in this project, we'll take a look at the criteria necessary for a set + operation to be a group and for a set + collection of subsets to be a topological space.

CHAPTER 3

Metric Spaces

A surface is more than just a set of points. Points on a surface have a notion of *closeness* that doesn't exist with a set unless we add some structure.

One way we can introduce the idea of closeness is to introduce the idea of the distance between points. That is, a function d that gives us a number for any pair of points.

To be a **distance function**, our function must meet some additional requirements:

- all distances must be non-negative: $d(x, y) >= 0$
- the distance between x and y is zero if and only if x and y are the same point: $d(x, y) = 0 \iff x = y$
- the distance between x and y is the same as the distance between y and x. In other words, the distance function is always *symmetric*: $d(x, y) = d(y, x)$
- finally, the distance between two points can never exceed the sum of the distance between each of the points and a third point. This is often referred to as the *triangle rule*: $d(x, z) <= d(x, y) + d(y, z)$

A distance function is often called a **metric**. A set of points with a distance function is called a **metric space**.

A metric space clearly has a notion of closeness. A point y is closer to x than z is if $d(y, x) < d(z, x)$.

CHAPTER 4

Open Balls and Continuity

Previously, we introduced the notion of a *Metric Spaces*.

Once you have a metric space (that is, a set of points and a function that specifies how distant you consider any two points) then you can start to develop notions of continuity and limits that form the basis for analysis.

For a metric space (X, d), let us call all the points less than r away from a point a the **open ball** of radius r at point a. In other words, $B(X, d, a, r) = \{x \in X | d(a, x) < r\}$.

We can then define continuity by saying that a function f between the metric spaces (X_1, d_1) and (X_2, d_2) is **continuous** at point a in X iff, given a positive r_2, there is a positive r_1 such that $f(B(X_1, d_1, a, r_1))$ is a subset of $B(X_2, d_2, f(a), r_2)$. In other words, f is continuous at point a if you can always provide an open ball at a that maps to points within an arbitrarily small radius open ball at $f(a)$.

Once you think of continuity in terms of open balls, you're able to do something interesting. Imagine that you don't know the distance function of either the domain or co-domain of the function but someone who does has precalculated all the open balls for you. Of course, for most metric spaces, there would be an infinite number of these, but the key point here is that you only need to know what the open balls are to test continuity. You don't need to know the distance function.

Let's call a set of points with the open balls precalculated an **open ball space**. Clearly it is easy to turn any metric space into an open ball space. You can't go the other way, however, as we've thrown out what the actual radius of each open ball is.

But we're now able to talk about continuity with a more general set structure than a metric space. There are many other notions that can be introduced on an open ball space, some of which we'll get to on our journey through the Poincaré Conjecture.

In the next Poincaré Project entry, however, we will take one more step of abstraction and get to the very core concept of topology itself.

CHAPTER 5

Open Sets

In *Open Balls and Continuity*, I said:

> Imagine that you don't know the distance function of either the domain or co-domain of the function but someone who does has precalculated all the open balls for you. Of course, for most metric spaces, there would be an infinite number of these, but the key point here is that you only need to know what the open balls are to test continuity. You don't need to know the distance function.

So imagine that a friend has given you a set along with all of the subsets that are open balls. From this alone, you can establish whether a function on the set is continuous.

We can simplify the definition of continuity (and other concepts) by introducing a more general subset than the open ball called an *open set*. Even though we will initially define open sets in terms of open balls, we can simply provide the open sets without reference to open balls, much like your friend provided the open balls without reference to the metric.

A subset of a set X is called an **open set** if it is the union of open balls of X.

Now continuity can be defined in terms of open sets (and this definition can be proven to be the same as that using open balls). A function from X to Y is continuous if and only if, for each open set in Y, the inverse in X is also an open set.

So, instead of giving you the open balls, your friend could give the open sets. And from that, you'd be able to establish whether a function was continuous.

Now you might be wondering: in the absence of the original metric, how would we know whether the collection of open sets was really a collection of open sets and not just some random selection of subsets? (Perhaps you don't trust your friend.) Well, it turns out open sets have some interesting properties:

- the union of any collection of open sets in X is also an open set in X;
- the intersection of any finite collection of open sets in X is also an open set in X;
- the empty set is open;
- the set X itself is open.

What is so significant about this is that a collection of subsets of X is a collection of open subsets of X if (and only if) it has the four properties above. It doesn't matter if it came from a distance function or open balls or some random selection of subsets. As long as the four properties above hold, the subsets are open subsets and can be used to demonstrate continuity (along with many other things).

NOTE: It is important that the union can be of any collection whereas the intersection can only be of a *finite* collection.

For a while, I thought the second two rules might be redundant and derivable from the first but a number of people (including Michael Walter and Richard Plagge) have clarified it for me. Michael points out the case where $X = \{1, 2\}$ and the candidate collection of sets is $\{\{1\}\}$. This meets the first two rules but not the second two. Richard gives the example $X = \{a, b, c, d\}$ with the candidate collection $\{\{a\}, \{a, b\}, \{a, b, c\}\}$. Again, the first two rules are met but the second two clearly do not follow.

CHAPTER 6

Topologies and Topological Spaces

We saw in *Open Sets* that open subsets of a set X always follow the rules:

- the union of any collection of open sets in X is also an open set in X;
- the intersection of any finite collection of open sets in X is also an open set in X;
- the empty set is open;
- the set X itself is open.

If you pick a collection of subsets of X that follows the four rules above, that collection is said to be a **topology on X**. Furthermore, a set along with a choice of topology on that set is called a **topological space**.

The use of the word *choice* is an important one. A given set will (unless it is a singleton) allow multiple valid topologies. It is the choice of topology that gives a topological space its characteristics rather than the the set itself.

Consider a simply set $\{a, b\}$. The smallest possible topology would be:

$\{\{\}, \{a, b\}\}$

In other words, the empty set and the the set itself are the only two open sets. This meets the definition of a topology and, in fact, for any set will be the smallest possible topology.

Another valid topology on $\{a, b\}$ would be:

$\{\{\}, \{a\}, \{b\}, \{a, b\}\}$

In other words, all subsets are open. This also meets the definition of a topology. For any set the topology which defines all subsets to be open will be the largest possible topology.

There are two other possible topologies that can be defined on the set $\{a, b\}$:

$\{\{\}, \{a\}, \{a, b\}\}$

and

$\{\{\}, \{b\}, \{a, b\}\}$

Step through the four rules to convince yourself that these are valid topologies for $\{a, b\}$.

Note that, although this example has involved a small, finite set, everything here applies to infinite sets too. It is possible to define, for example, different topologies on the set of real numbers. One such topology is one that equates the open intervals with the open sets. This is by far the most intuitive topology on the reals but by no means the only one.

CHAPTER 7

Injections, Surjections and Bijections

Imagine a school dance. There is a set of boys and a set of girls. When the music starts, each boy picks a girl to dance with.

Think of this as a mapping from a boy to a girl, or from an element in the set of boys to an element in the set of girls.

The mapping is said to be **injective** (or **one-to-one**) if each boy picks a different girl. If two boys try to dance with the same girl, the mapping isn't injective.

The mapping is said to be **surjective** (or **onto**) if no girls are left without a partner. If there is a girl not dancing, the mapping isn't surjective.

If the mapping is both injective and surjective it is said to be **bijective**.

You can immediately tell if there are the same number of boys and girls if the mapping is bijective—in other words, each boy is dancing with one and only one girl and no girls are left without a boy to dance with.

The existence of a bijection can be used to demonstrate that two sets have same number of elements or, in the case of infinite sets, have the same cardinality.

Bijections are also very important in establishing the equivalence between two structured sets (for example between two topological spaces) as we shall see in the near future.