Neutrino Documentation

Release Pi

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Contents

Neutrino is one of the most interesting particles in our world. The first proposal of such a new particle was given by Pauli. He managed to explain the spectrum of beta decay. In 1956, neutrinos was first detected in Cowan–Reines neutrino experiment.[1] Later on a lot of neutrino experiments have been carried out.

Solar Neutrino Problem

The sun produce neutrinos inside it and the neutrinos propagate out. On the earth we can detect them. The problem was that the detected neutrinos was only one third of the total neutrino flux predicted which causes some people to think that the solar had shut down. The solution, however, is the neutrino oscillation.

As far as we know, we have three flavours of neutrinos and their anti particles and they are orthogonal to each other,

$$\begin{aligned}\langle \nu_{l'} \mid \nu_l \rangle &= \delta_{l'l} \\ \langle \bar{\nu}_{l'} \mid \bar{\nu}_l \rangle &= \delta_{l'l} \\ \langle \bar{\nu}_{l'} \mid \nu_l \rangle &= 0.\end{aligned}$$

The interesting thing about neutrinos is that it oscillates.

Table of Contents:

[1] Cowan–Reines neutrino experiment

CHAPTER 1

Preliminary

This chapter is about the preliminary knowledge required by this topic.

Reactions Related to Neutrinos

1. Beta decays, $n \to p + e^- + \bar{\nu}$ and $p \to n + e^+ + \nu$
2. Electron capture and positron capture, $e^- + p \to n + \nu$ and $e^+ + n \to p + \bar{\nu}$.
3. Inverse beta decays, $\nu + n \to p + e^-$ and $\bar{\nu} + p \to n + e^+$.
4. Inverses of beta decays, $\bar{\nu} + e^- + p \to n$ and $n + e^+ + \nu \to n$.

What is a Neutrino Particle?

As Wigner said, a physical particle is an irreducible representation of the Poincaré group. A characteristic of Poincaré group is that mass comes in.

A neutrino particle is better recognized as its mass eigenstate.

In QFT, there are 3 different forms of neutrino mass term, left-handed Majorana, right-handed Majorana and Dirac mass terms.

Chirality and Helicity

Helicity

Helicity is the projection of spin onto direction of momentum,

$$h = \vec{J} \cdot \hat{p} = \vec{L} \cdot \hat{p} + \vec{S} \cdot \hat{p} = \vec{S} \cdot \hat{p},$$

where

$$\hat{p} = \frac{\vec{p}}{|\vec{p}|}$$

A state is called **right-handed** if helicity is positive, i.e., spin has the same direction as momentum.

Chirality

Chirality is the eigenstate of the Dirac γ_5 matrix, which is explicitly,[1]

$$\begin{aligned}\gamma^5 &= \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 01 & \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.\end{aligned}$$

Majorana or Dirac

Double Beta Decay

States

Wigner Function

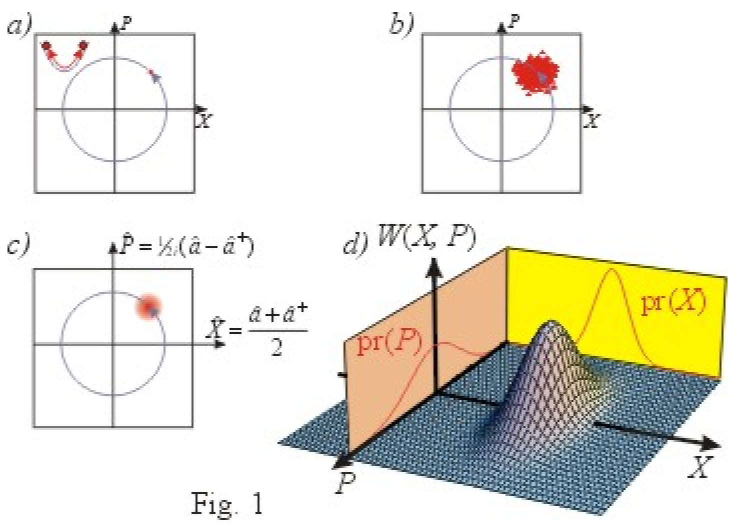

Fig. 1.1: A ensemble of classical harmonic oscillators can be described using such phase-space probability distribution.

Wigner function is an analogue of the classical phase-space probability distribution function though it is not really probability.[3] The mean of Wigner function lies in the two quadratures, i.e., space distribution and momentum distribution.

[1] *Chirality and Helicity In Depth* by Robert D. Klauber

[3] http://www.iqst.ca/quantech/wigner.php

There is a collection of Wigner functions on this site.[3]

.. admonition:: Question

class warning

How do one describe a system of neutrinos using Wigner function? What is the effect of statistics.

Statistics

Fermi-Dirac distribution

$$f(p,\xi) = \frac{1}{1+\exp(p/T-\xi)},$$

where $\xi = \mu/T$ is the degeneracy parameter.

The neutrino-neutrino forward scattering is[2]

$$\nu_\alpha(p) + \nu_\beta(k) \rightarrow \nu_\alpha(k) + \nu_\beta(p)$$

Question

Meaning of each term in Liouville equation ?

Refs & Notes

[2] Pantaleone (1992), Friedland & Lunardini (2003).

CHAPTER 2

Common Sense

Units

Natural units makes the calculation of neutrinos convinient. The consequences are

1. The energy-mass-momentum relations becomes $E^2 = p^2 + m^2$. Thus mass m, momentum $\mathbf{p}$ and energy E have the same units.
2. Angular momentum in quantum mechanics is $L_z = m\hbar$ where m is a number. $\hbar$ is of unit angular momentum.
3. A plane wave in quantum mechanics is $\Psi = Ae^{\frac{Et-px}{\hbar}}$. $\frac{Et-px}{\hbar}$ should be unitless, which means px has unit angular momentum, which is obvious, while Et also has the unit of angular momentum. Previously we noticed momentum has the same unit with energy, we should have time t has the same unit as length x. Also we can conclude that length and time has the unit of $1/E$.

One should notice that charge is unit 1 in natural units since

$$F = \frac{Qq}{4\pi r^2}.$$

The conversion between natural units and SI can be down by using the following relations.

$$1\mathrm{GeV} = 5.08 \times 10^{15}\mathrm{m}^{-1}$$
$$1\mathrm{GeV} = 1.8 \times 10^{-27}\mathrm{kg}$$

CHAPTER 3

Mathematics Related

The Equations

For 2 flavor oscillations, the equation for flavor neutrinos is

$$i\frac{d}{dt}\begin{pmatrix}\nu_e\\ \nu_x\end{pmatrix} = \frac{\Delta m^2}{4E}\begin{pmatrix}-\cos 2\theta & \sin 2\theta\\ \sin 2\theta & \cos 2\theta\end{pmatrix}\begin{pmatrix}\nu_e\\ \nu_x\end{pmatrix}$$

and with matter

$$i\frac{d}{dt}\begin{pmatrix}\nu_e\\ \nu_x\end{pmatrix} = \frac{\Delta m^2}{4E}\begin{pmatrix}\frac{4E}{\Delta m^2}\sqrt{2}G_F n_e - \cos 2\theta & \sin 2\theta\\ \sin 2\theta & \cos 2\theta\end{pmatrix}\begin{pmatrix}\nu_e\\ \nu_x\end{pmatrix}$$

Qualititative Analysis

The vacuum oscillation is determined by autonomous equations. A fixed point of an autonomous system is defined by

$$\frac{d}{dt}\begin{pmatrix}\nu_e\\ \nu_x\end{pmatrix} = 0,$$

which means the so called "velocity" is 0. For vacuum oscillation, we set

$$\begin{pmatrix}-\cos 2\theta & \sin 2\theta\\ \sin 2\theta & \cos 2\theta\end{pmatrix}\begin{pmatrix}\nu_e\\ \nu_x\end{pmatrix} = 0.$$

Thus we find the fixed points,

$$\begin{aligned}\nu_e &= 0\\ \nu_x &= 0.\end{aligned}$$

If we have only the *i*th function with derivative 0, the line is called the *i*th-nullcline. Thus the fixed points are the interaction points of all the nullclines.

These fixed points are very useful. In general, for a set of autonomous equations,

$$\begin{aligned} f'(x) &= F(f,g) \\ g'(x) &= G(f,g), \end{aligned}$$

by definition the fixed point in phase space $\{f_i, g_i\}$ leads to the result

$$\begin{aligned} F(f,g) &= 0 \\ G(f,g) &= 0. \end{aligned}$$

Thus the equations can be approximated using Taylor expansion near the point $\{f_i, g_i\}$, since at the fixed points the derivatives are small.

$$\begin{aligned} \frac{d}{dx} &= F(f,g) \\ &= F(f_i,g_i) + \frac{\partial F(f,g)}{\partial f}|_{f=f_i,g=g_i}(f-f_i) + \frac{\partial F(f,g)}{\partial g}|_{f=f_i,g=g_i}(g-g_i) + \mathcal{O}(2). \end{aligned}$$

The equations are simplified to linear equations whose coefficient matrix is simply the Jacobian matrix of the original system at the fixed point $\{f_i, g_i\}$. In this example, the coefficient matrix for the linearized system is

$$\mathbf{C} = \begin{pmatrix} \frac{\partial F(f,g)}{\partial f}|_{f=f_i,g=g_i} & \frac{\partial F(f,g)}{\partial g}|_{f=f_i,g=g_i} \\ \frac{\partial G(f,g)}{\partial f}|_{f=f_i,g=g_i} & \frac{\partial G(f,g)}{\partial g}|_{f=f_i,g=g_i} \end{pmatrix}.$$

As a comparison, the Jacobian matrix for the orginal equations at the fixed point is also the same which quite makes sense because Jacobian itself is telling the first order approximation of the velocity.

This linearization is only valid for hyperbolic fixed points which means that the eigenvalues of Jacobian matrix at fixed point has non-zero real part. Suppose the Jacobian is $\mathbf{J}$ with eigenvalues are λ_j, a hyperbolic fixed point requires that $\mathcal{R}\rceil\lambda_j \neq 0$.

For more analysis, checkout *Poincare-Lyapunov Theorem*.[1]_

Define $p = \mathrm{Tr}(\mathbf{J}(f_i, g_i))$ and $q = \det(\mathbf{J}(f_i, g_i))$ then the systems can be categorized into 3 different categories given the case that the fixed point isa hyperbolic one.

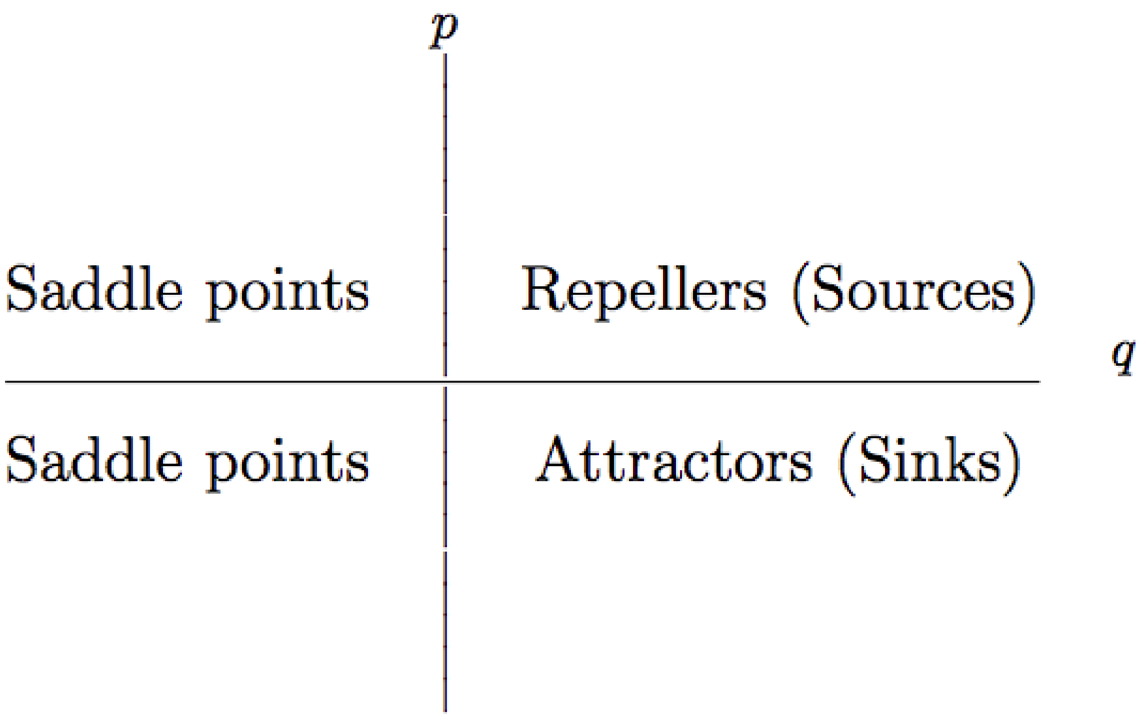

Fig. 3.1: A diagram that shows the different categorizations given p and q values. Repellers and saddle points are unstable points but attractors are stable. Or in simple ways, given the eigenvalues of the Jacobian λ_1, λ_2, $Re(\lambda_1) > 0, Re(\lambda_2) > 0$ gives us a repeller, $Re(\lambda_1) < 0, Re(\lambda_2) < 0$ gives us an attractor while $Re(\lambda_1) < 0, Re(\lambda_2) > 0$ gives us the saddle point.

CHAPTER 4

Masses of Neutrinos

Neutrino masses are still not determined completely. However we have some possible patterns.

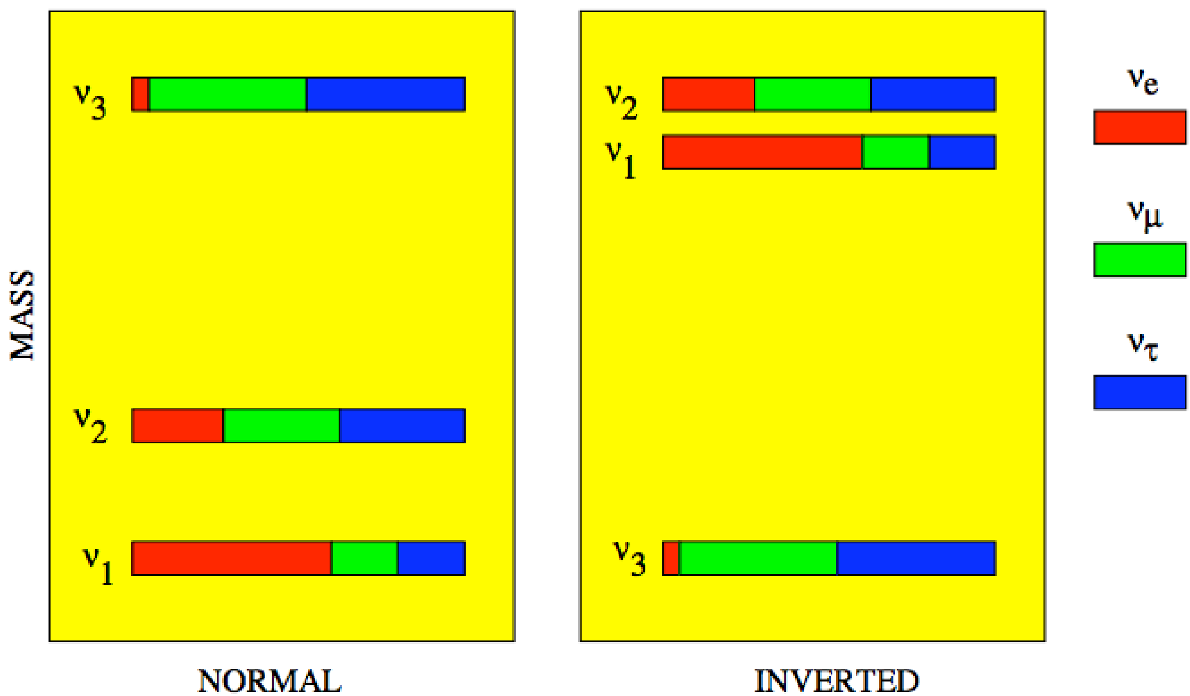

Fig. 4.1: Source: http://projects.fnal.gov/nuss/lectures/RabiM_1.pdf

One of the questions we have about the masses of neutrinos is **the generation of it**.

Note: This figure also gives the terms: normal hierarchy (NH) and invertd hierarchy (IH).

Lepton mixing matrix, can be written as the product of three matrices which stands for a rotation in 23, 13(with a CP phase), 12 respectively. This is called the PMNS mixing matrix.

$$
\begin{aligned}
\mathbf{U} &= \mathbf{U_{23}} \times \mathbf{U_{13}}_{,\delta} \times \mathbf{U_{12}} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 - \sin\theta_{23} & \cos\theta_{23} & \end{pmatrix} \begin{pmatrix} \cos\theta_{13} & 0 & e^{i\delta}\sin\theta_{13} \\ 0 & 1 & 0 \\ -e^{i\delta}\sin\theta_{13} & 0 & \cos\theta_{13} \end{pmatrix} \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

See-saw Mechanism

RH neutrinos term in Lagrangian breaks the symmetry.

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CHAPTER 5

How Do Neutrinos Propagate

Question

How to interpret neutrino propagation and scattering using wave packet formalism?

In the book of *Principles of Quantum Mechanics*, Shankar shows how to deal with scattering using just wave packet.

What I can do is to check the following questions.

1. How do wave packet formalism help us understanding the scattering of neutrinos.
2. How do relativistic case change the results?
3. What if the packet is a combination of Gaussian packets?

Wave Packet Treatment

From uncertainty principle we know it's not good enough to treat neutrinos as mono-momentum particles because our measurement measures the momentum with an accuracy and the position of the neutrinos are not completely determined. We have both momentum width and position width which looks a lot like a wave packet.

The caveats are

1. What are the energies, momenta, velocities of neutrinos and the average of them?
2. How to find the amplitude of wave packet? What's the geometry of the wave packet?
3. The time evolution should reduce to the single particle formalism in some limits.

In principle we need all the information about the generation of neutrinos. However, we can use some unknown paramters to derive the formalism of the wave packets then investigate the unknown paramters.

A wave packet is constructed with a distribution of amplitude at each momentum and position and time.

Note: A wave packet in wave dynamics is bunch of plane waves that makes a localized packet. For example one of the general form of wave packets is

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{i(kx-\omega(k)t)}\, dk.$$

Basically, one needs a lot of frequencies/wavenumbers/momenta to construct some localized waves.

As an application of this general wave packet, we can write down the wave packet of neutrinos using an assumed initial distribution over all possible momenta. **The problem is that we have no idea what the amplitude should be.**

Some Questions

Some questions should be answered in this formalism.

1. What are ν_f and ν_m in this formalism?

Question

What is ν_f, i.e., the flavour state, in the formalism of wave packet?

Answer

In the view of math, the flavour state is a superposition of all mass states,

$$\psi(x,t) = \int_{lower}^{upper} dp'_\nu \sum_m U_{fm} a(p_\pi^m(p'_\nu))\nu_m e^{ip'_\nu x} e^{-iE_m(p'_\nu)t}$$

In other words, as long as we can measure the wave packet in a sense that the position difference is large enough, the wave packet still.

Question

What does decoherence mean then?

Answer

An first idea can be that the wave packets of different mass eigen states are travelling at different speed thus they get very far apart after some travelling time.

However we should be careful with the wave packet formalism. This treatment is infact an effective treatment in my understanding, to reconcile the fact that the neutrinos are actually not at a definite position and momentum state due to quantum uncertainty principle.

So any discussion about the decoherence of the wave packets should make clear of the measurements including the production procedure.

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CHAPTER 6

Oscillations - In General

Evidence of Oscillations

A lot of experiments have been done to research on neutrino oscillations. In summary there are three types,

1. Solar neutrinos,
2. Reactor and accelerator neutrinos,
3. Atmospheric neutrinos.

Results of Experiments

1. Difference between masses from data

$$\frac{|\Delta m_{21}^2|}{|\Delta m_{31(32)}^2|} \approx 0.03.$$

   We also have

$$|\Delta m_{21}^2| \ll |\Delta m_{31(32)}^2|.$$

   By some convention, people would use numbers so that $\Delta m_{21}^2 > 0$ or $m_1 < m_2$.

Determine $|\Delta m^2|$ and θ

The neutrino experimantal data shows the mixing angles are[1]

1. $\theta_{23} = 39^\circ \pm 2^\circ$;
2. $\theta_{13} = 8.9^\circ \pm 0.5^\circ$;
3. $\theta_{12} = 34^\circ \pm 1^\circ$.

[1] Neutrino tomography by Margaret A. Millhouse & David C. Latimer, American Journal of Physics 81, 646 (2013); doi: 10.1119/1.4817314 .

Experimental result of the :math:‘ delta m^2 _{ij}‘s are[1]

1. $\delta^2 m_{21} = 7.5^{+0.3}_{-0.2} \times 10^{-5} eV^2$;
2. $|\delta^2 m_{32}| = 2.4^{+0.1}_{-0.1} \times 10^{-3} eV^2$.

Definition of Mass-squared Difference

:math:‘ delta m^2 _{ij}=m_i^2-m_j^2‘. Obviously, $\delta^2 m_{31} = \delta^2 m32 - \delta^2 m_{21}$.

As $|\delta^2 m_{21}| \ll |\delta^2 m_{32}|$, we should have $\delta^2 m_{31} \approx \delta^2 m_{32}$.

Atmospheric Results

Accelerator Results

Reactor Results

Vacuum Theory

Neutrinos evolve in mass eigenstates. So we need to describe flavour states $|\nu_\alpha\rangle$ using mass eigenstates $|\nu_j\rangle$.

$$|\nu_\alpha\rangle = \sum_j U^*_{\alpha j} |\nu_j; \tilde{p}_j\rangle ,$$

where $U^*_{\alpha j}$ is the element of neutrino mixing matrix.

PMNS Mixing Matrix

Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix is the product of three rotation matrices, in addition to an extra phase,

$$\begin{aligned}\mathbf{U} &= \mathbf{U_{23}} \times \mathbf{U_{13,\delta}} \times \mathbf{U_{12}} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 - \sin\theta_{23} & \cos\theta_{23} & \end{pmatrix} \begin{pmatrix} \cos\theta_{13} & 0 & e^{i\delta}\sin\theta_{13} \\ 0 & 1 & 0 \\ -e^{i\delta}\sin\theta_{13} & 0 & \cos\theta_{13} \end{pmatrix} \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}\end{aligned}$$

The δ is the CP violation phase.

The origin of the phase is from the fact that we need 4 degrees of freedom for this mixing matrix while a convinient way is to write down the SO(3) rotation matrix then put this extra phase here.

More About Phase of Nutrinos

The mixing of mass eigenstates is

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & e^{i\alpha_3} \end{pmatrix} \text{Some Unitary Matrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & e^{i\beta_3} \end{pmatrix}$$

Since the phase of neutrinos can be redefined, we have 3 phases for each flavour and a global phase being arbitary. The first matrix on the RHS can be eliminated. **The third matrix on the RHS is not important for neutrino oscillations so it can be neglected.** (Proof required)

In ultra relativistic case, we can simply find out the time evolution, which is equivalent to distance evolution,

$$|\psi(t)\rangle = \sum_j U^*_{\alpha j} G_j(t,t_0) \left|\nu_j;\tilde{p}_j\right\rangle.$$

The survival probability means how much neutrinos of a flavour left after some time or distance, which is calculated by

$$P(\nu_l \to \nu_{l'}) = |\langle \nu_{l'} \mid \psi(t)\rangle|^2.$$

We can see clearly that the survival probability depends on some parameters.

Two Flavour Oscillation

To write down this clearly, we need to write down the mixing matrix and propagator. For simplicity, we calculate the example of two flavour (a, b) oscillation.

It's easier to write down the propagation in mass eiginstates so the first thing to work out is the mixing matrix.

Suppose we have only a flavour neutrino initially,

$$|\psi(0)\rangle = |\nu_a\rangle$$

Mixing Matrix

The mixing matrix is an rotation of eigenbasis.

The flavour states can be expressed in terms of mass eigenstates,

$$\begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

where the matrix

$$\mathbf{U} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

is the mixing matrix which is a rotation of basis geometrically. In other words, this matrix is the representation of the rotation $e^{i\hat{\theta}}$.

Survival Probability

With the mixing matrix, the propagation of an initial state of only flavour a is

$$|\psi(t)\rangle = \cos\theta\,|\nu_1\rangle\, e^{-iE_1 t} + \sin\theta\,|\nu_2\rangle\, e^{-iE_2 t}.$$

To find out the amplitude of flavour a, we need to project the state $|\psi(t)\rangle$ onto a flavour eigenstate, say, $|\nu_a\rangle$,

$$\begin{aligned}\langle \nu_a \mid \psi(t)\rangle &= \langle \nu_a| \left(\cos\theta\,|\nu_1\rangle\, e^{-iE_1 t} + \sin\theta\,|\nu_2\rangle\, e^{-iE_2 t}\right) \\ &= \left(\cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle\right)\left(\cos\theta\,|\nu_1\rangle\, e^{-iE_1 t} + \sin\theta\,|\nu_2\rangle\, e^{-iE_2 t}\right) \\ &= \cos^2\theta e^{-iE_1 t} + \sin^2\theta e^{-iE_2 t}\end{aligned}$$

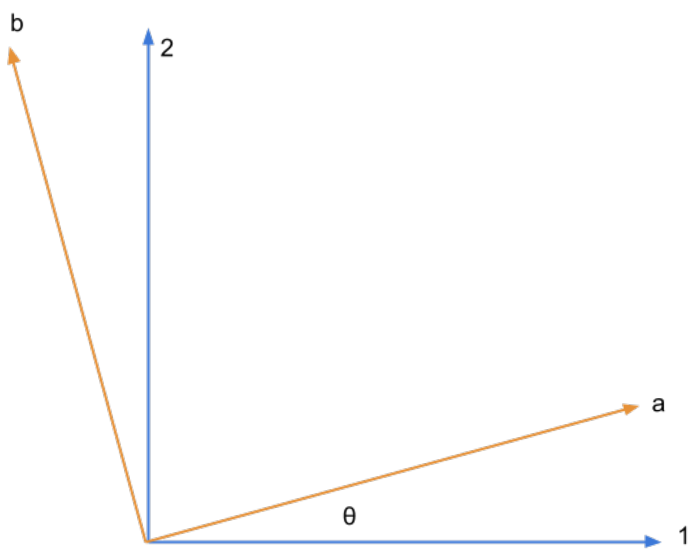


Fig. 6.1: Two flavour neutrino mixing diagram with θ being the mixing angle

The survival probability is the amplitude squared,

$$\begin{aligned}
P_{aa} &= |\langle \nu_a \mid \psi(t)\rangle|^2 \\
&= |\cos^2\theta e^{-iE_1 t} + \sin^2\theta e^{-iE_2 t}|^2 \\
&= \left(\cos^2\theta e^{-iE_1 t} + \sin^2\theta e^{-iE_2 t}\right)^* \left(\cos^2\theta e^{-iE_1 t} + \sin^2\theta e^{-iE_2 t}\right) \\
&= \cos^4\theta + \sin^4\theta + \cos^2\theta\sin^2\theta e^{i(E_1-E_2)t} + \sin^2\theta\cos^2\theta e^{-i(E_1-E_2)t} \\
&= \cos^4\theta + \sin^4\theta + \cos^2\theta\sin^2\theta e^{i\Delta E t} + \sin^2\theta\cos^2\theta e^{-i\Delta E t} \\
&= \cos^4\theta + \sin^4\theta + 2\cos^2\theta\sin^2\theta\cos(\Delta E t) \\
&= (\cos^2\theta + \sin^2\theta)^2 - 2\cos^2\theta\sin^2\theta + 2\cos^2\theta\sin^2\theta\cos(\Delta E t) \\
&= 1 - 2\cos^2\theta\sin^2\theta(1-\cos(\Delta E t)) \\
&= 1 - \sin^2(2\theta)\sin^2\left(\frac{\Delta E t}{2}\right)
\end{aligned}$$

with the definition $\Delta E = E_1 - E_2 \approx p_1 + \frac{1}{2}\frac{m_1^2}{p_1} - p_2 - \frac{1}{2}\frac{m_2^2}{p_2}$. We usually calculate the case $p_1 = p_2 = p$, which takes us to

$$\begin{aligned}
\Delta E &\approx \frac{m_1^2 - m_2^2}{2p} \\
&= \frac{\delta^2 m}{2p}.
\end{aligned}$$

with $\delta^2 m = m_1^2 - m_2^2$. Most of the time we would like to know the oscillation with respect to distance. Using the approximation $t = L$ and $\Delta E \approx \frac{m_1^2 - m_2^2}{2p}$, we have

$$\begin{aligned}
P_{aa} &= 1 - \sin^2(2\theta)\sin^2\left(\frac{\Delta E L}{2}\right) \\
&= 1 - \sin^2(2\theta)\sin^2\left(\frac{\delta m^2 L}{4p}\right).
\end{aligned}$$

This is the survival probability of flavour a neutrino with an initial state of flavour a.

There are several things to be noticed,

1. $\theta = 0$ leads to oscillation free neutrinos.
2. $\Delta E = 0$ or $\delta^2 m = 0$ (in the case of same momentum) also gives us no oscillation.
3. At $L = 0$ the survival probability is 1, which means no oscillation is done.

Hamiltonian

It's easy to write down the Hamiltonian for the mass state stationary Schrödinger equation. As we have proven, to first order approximation,

$$E = p + \frac{1}{2}\frac{m^2}{p}$$

$$\begin{aligned}
\mathbf{H}_j &= \begin{pmatrix} p + \frac{1}{2}\frac{m_1^2}{p} & 0 \\ 0 & p + \frac{1}{2}\frac{m_2^2}{p} \end{pmatrix} \\
&= p\mathbf{I} + \frac{1}{2p}\begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}
\end{aligned}$$

However, the Hamiltonian we prefer is the one for flavour eigenstates. To achieve this, we only need to rotate this previous Hamiltonian using the mixing matrix $\mathbf{U}$.

$$\begin{aligned}
\mathbf{H}_\alpha &= \mathbf{U}\hat{H}_j\mathbf{U}^T \\
&= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \left(p\mathbf{I} + \frac{1}{2p} \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} \right) \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\
&= p\mathbf{I} + \frac{1}{2p} \begin{pmatrix} \cos^2\theta m_1^2 + \sin^2\theta m_2^2 & -\sin\theta\cos\theta m_1^2 + \sin\theta\cos\theta m_2^2 \\ -\sin\theta\cos\theta m_1^2 + \sin\theta\cos\theta m_2^2 & \sin^2\theta m_1^2 + \cos^2\theta m_2^2 \end{pmatrix} \\
&= p\mathbf{I} + \frac{1}{2p} \begin{pmatrix} m_1^2 - \delta^2 m \sin^2\theta & -\frac{1}{2}\sin 2\theta\delta m^2 \\ -\frac{1}{2}\sin 2\theta\delta m^2 & m_2^2 + \delta m^2 \sin^2\theta \end{pmatrix} \\
&= p\mathbf{I} + \frac{1}{2p} \left(\frac{1}{2}(m_1^2 + m_2^2)\mathbf{I} - \frac{1}{2} \begin{pmatrix} -\delta m^2 \cos 2\theta & \delta^2 m \sin 2\theta \\ \delta m^2 \sin 2\theta & \delta^2 m \cos 2\theta \end{pmatrix} \right) \\
&= \left(p + \frac{m_1^2 + m_2^2}{4p} \right) \mathbf{I} - \frac{1}{4p} \begin{pmatrix} -\delta m^2 \cos 2\theta & \delta^2 m \sin 2\theta \\ \delta m^2 \sin 2\theta & \delta^2 m \cos 2\theta \end{pmatrix}
\end{aligned}$$

Again we see clearly, no oscillation will apear as long as mixing angle $\theta = 0$ or $\delta m^2 = 0$.

Note: The reason we can do this is that this mixing matrix is time and space independent. To see this, we first write down the Schrödinger equation for mass eigenstates,

$$i d_t \left|\Phi_j\right\rangle = \hat{H}_j \left|\Phi_j\right\rangle .$$

Applying the mixing matrix,

$$i d_t \mathbf{U}^{-1} \left|\Phi_\alpha\right\rangle = \hat{H}_j \mathbf{U}^{-1} \left|\Phi_\alpha\right\rangle .$$

Notice that the mixing matrix, which is a rotation, is orthonormal, $\mathbf{U}\mathbf{U}^T = \mathbf{I}$. Then we have inverse of this matrix is the same as the transpose.

$$i d_t \mathbf{U}^T \left|\Phi_\alpha\right\rangle = \hat{H}_j \mathbf{U}^T \left|\Phi_\alpha\right\rangle .$$

Multiply on both sides $\mathbf{U}$ and remember the fact that the mixing matrix is orthonormal, we have

$$i d_t \left|\Phi_\alpha\right\rangle = \mathbf{U}\hat{H}_j \mathbf{U}^T \left|\Phi_\alpha\right\rangle .$$

Now we can define the Hamiltonian for flavour states,

$$\mathbf{H}_\alpha = \mathbf{U}\mathbf{H}_j\mathbf{U}^T.$$

Since Pauli matrices plus identity forms a complete basis for all 2 by 2 matrices, it our Hamiltonian can be written as

$$\begin{aligned}
\mathbf{H} &= \frac{\delta^2 m}{4E} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \\
&= \frac{\delta^2 m}{4E} \left(-\cos 2\theta \sigma_{\mathbf{z}} + \sin 2\theta \sigma_{\mathbf{x}} \right) .
\end{aligned}$$

Note: Pauli matrices are

$$\begin{aligned}
\sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\sigma_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
\sigma_x &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .
\end{aligned}$$

In a more compact way,

$$\sigma_j = \begin{pmatrix} \delta_{j3} & \delta_{j1} - i\delta_{j2} \\ \delta_{j1} + i\delta_{j2} & -\delta_{j3} \end{pmatrix}.$$

Equation of Motion in Matter

Hamiltonian

We have already derived the Hamiltonian for vacuum oscillatioin,

$$H_v = \frac{\delta m^2}{2E}\frac{1}{2}\begin{pmatrix} -\cos 2\theta_v & \sin 2\theta_v \\ \sin 2\theta_v & \cos 2\theta_v \end{pmatrix},$$

where we would like to define a new matrix,

$$\mathbf{B} = \frac{1}{2}\begin{pmatrix} -\cos 2\theta_v & \sin 2\theta_v \\ \sin 2\theta_v & \cos 2\theta_v \end{pmatrix},$$

so that the vacuum Hamiltonian can be written as

$$H_v = \frac{\delta m^2}{2E}\mathbf{B}$$

The **effect of matter**, as we have already discussed before, adds an extra term

$$H_m = \sqrt{2} G_F n_e L.$$

Here we have

$$L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Note: Previously in the MSW effect section, we have $L = \frac{1}{2}\sigma_3$. The reason, as explained there, is that we can always write down a 2 by 2 matrix using Pauli matrices and indentity matrix and identity matrix only shifts the overall eigenvalue not the eigenvector so we can just drop the identity term.

One other term is the self-interaction of neutrinos, i.e., neutral-current neutrino-neutrino forward exchange scattering,

$$H_\nu = \sqrt{2} G_F \int d^3\mathbf{p}' (1 - \hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')(\rho_{p'} - \bar{\rho}_{p'}).$$

The overall Hamiltonian is

$$H = H_0 + H_m + H_\nu,$$

where the vacuum Hamiltonian is

$$\begin{aligned} H_0 &= \frac{\delta^2 m}{2E}\mathbf{B} \\ &= \frac{\delta^2 m}{2E} U \left(\frac{1}{2}\sigma_3\right) U^\dagger. \end{aligned}$$

Equation of Motion

From the Hamiltonian, Von Neumann equation is

$$i\frac{\partial}{\partial t}\rho = [H,\rho]$$

In Picture chapter we have seen the definition of a polarization matrix. The components of a polarization vector (**for neutrinos**) is given by

$$\begin{aligned} P_{\omega,i} &\propto \mathrm{Tr}(\rho_E\sigma_i) \\ &= \frac{1}{n_\nu}\frac{|\delta^2 m|}{2\omega^2}\times \mathrm{Tr}(\rho_E\sigma_i). \end{aligned}$$

For anitneutrinos, we have a negative ω which is defined as $\omega = \frac{\delta m^2}{2E}$ (neutrinos) and $\omega_{\bar{\nu}} = -\frac{\delta m^2}{2E}$ (anitneutrinos). The polarization is defined as

$$P_{\omega,i} = -\frac{1}{n_\nu}\frac{|\delta^2 m|}{2\omega^2}\times \mathrm{Tr}(\bar{\rho}_E\sigma_i).$$

With all these definitions, Von Neumann equation multiply by $\vec{\sigma} = \sigma_1\hat{e}_1 + \sigma_2\hat{e}_2 + \sigma_3\hat{e}_3$, we have

$$i\dot{\rho}\sum_i \sigma_i\hat{e}_i = [H,\rho]\sum_i \sigma_i\hat{e}_i.$$

Notice that Pauli matrices are Hermitian and Unitary, we can alway insert the identity $\mathbf{I} = \sigma_j\sigma_j^\dagger$.

Commutator and Cross Product

Commutator of two vectors,

$$\vec{A}\times\vec{B} = (A_2B_3 - A_3B_2)\hat{e}_1 + (A_3B_1 - A_1B_2)\hat{e}_2 + (A_1B_2 - A_2B_3)\hat{e}_3$$

Trace of Pauli Matrices

All Pauli matrices have vanishing trace. And what makes our calculation more convinient is that the trace of matrices is invariant under cyclic permutation, that is

$$\mathrm{Tr}(\sigma_i\mathbf{H}\sigma_j) = \mathrm{Tr}(\mathbf{H}\sigma_j\sigma_i)$$

Notice that to have a non-vanishing trace we need $i = j$. This property really saves our life.

As the definition, we have

$$\begin{aligned} \mathbf{H} &= \vec{H}\cdot\vec{\sigma} \\ \rho &= \vec{\rho}\cdot\vec{\sigma} \end{aligned}$$

Using these we can rewrite the commutator

$$
\begin{aligned}
[H,\rho] &= [\vec{H}\cdot\vec{\sigma},\vec{\rho}\cdot\vec{\sigma}] \\
&= \sum_{ik}(H_i\sigma_i\rho_k\sigma_k - \rho_k\sigma_k H_i\sigma_i) \\
&= \sum_{ik}(H_i\rho_k\sigma_i\sigma_k - \rho_k H_i\sigma_k\sigma_i) \\
&= \sum_{ik} H_i\rho_k(\sigma_i\sigma_k - \sigma_k\sigma_i) \\
&= \sum_{ik} H_i\rho_k[\sigma_i,\sigma_k] \\
&= \sum_{ik} H_i\rho_k 2i\epsilon_{ikn}\sigma_n \\
&= 2i\sum_{ik}\epsilon_{ikn}\sigma_n H_i\rho_k
\end{aligned}
$$

Multiply by σ_j and take the trace, we get,

$$
\begin{aligned}
\mathrm{Tr}(\sigma_j[H,\rho]) &= 2i\mathrm{Tr}(\sum_{ik}\epsilon_{ikn}\sigma_j\sigma_n H_i\rho_k) \\
&= 2i\sum_{ik}\mathrm{Tr}(\epsilon_{ikj}\mathrm{I}H_i\rho_k) \\
&= 2i\sum_{ik}\epsilon_{jik}H_i\rho_k\mathrm{Tr}(\mathbf{I}) \\
&= 4i\epsilon_{jik}H_i\rho_k.
\end{aligned}
$$

The corresponding LHS after these work becomes

$$
\begin{aligned}
i\mathrm{Tr}(\sigma_j\dot{\rho}_i\sigma_i) &= i\partial_t\rho_j\mathrm{Tr}(I) \\
&= 2i\dot{P}_j
\end{aligned}
$$

The Von Neuman equation becomes

$$
\dot{\vec{P}} = 2\vec{H}\times\vec{P}.
$$

We know explicitly what polarization vector is

$$
P_j = \mathrm{Constant}\mathrm{Tr}(\rho\sigma_j)
$$

for neutrinos while

$$
\bar{P}_j = -\mathrm{Constant}\mathrm{Tr}(\bar{\rho}\sigma_j).
$$

The vectorized Hamiltonian is

$$
H = H_i\sigma_i.
$$

Multiply by σ_j and take the trace,

$$
\mathrm{Tr}(H\sigma_j) = H_j\mathrm{Tr}(\mathbf{I}),
$$

that is,

$$
\mathrm{Tr}(H\sigma_j) = 2H_j.
$$

Hamiltonian

The Hamiltonian for homogeneous isotropic environment is

$$\begin{aligned} H &= H_0 + H_m + H_\nu \\ &= \omega \mathbf{B} + \lambda \mathbf{L} + \sqrt{G_F} \int_0^\infty dE' (\rho'_E - \bar{\rho}'_E). \end{aligned}$$

Then the equation we need becomes

$$\dot{\vec{P}}_\omega = (\omega \vec{B} + \lambda \vec{L} + \mu \vec{D}) \times \vec{P}_\omega.$$

where $\vec{B} = \mathrm{Tr}(\mathbf{B}\vec{\sigma})$, $\vec{L} = \mathrm{Tr}(\mathbf{L}\vec{\sigma})$, $\vec{D} = \int_{-\infty}^{\infty} d\omega \vec{P}_\omega$.

Q&A

Question

What are some of the conventions used in liturature?

Answer

1. $\Delta m_{ij}^2 = m_i^2 - m_j^2$.
2. Flavours of left hand neutrinos are mixing of mass eigen states, $\nu_{lL} = \sum_{j=1}^{3} U_{lj} \nu_{jL}(x)$.

Question

Why can we use just quantum mechanics on relativistic neutrinos? In principle one should use quantum field theory or at least relativistic quantum mechanics?

Answer

To be answered.

Question

What does the mixing angle mean exactly both in vacuum and matter environment?

Answer

There are several ways to illustrate this.

1. **Rotation angle** in flavour space. For simplicity I use a two component neutrino model.

$$
\begin{aligned}
|\nu_1\rangle &= \cos\theta\,|\nu_e\rangle + \sin\theta\,|\nu_\mu\rangle \\
|\nu_2\rangle &= -\sin\theta\,|\nu_e\rangle + \cos\theta\,|\nu_\mu\rangle
\end{aligned}
$$

This is a rotation in a plane with a generator $e^{-i\hat{\theta}}$. **(Make a figure for this.) + (Write down the 3 components model.)**

2. **Oscillation probability** involves this angle too. It is a suppression of the oscillation probability.
3. From the view of **quantum states**, this angle determines how the flavour states are composed with mass eigenstates, i.e., the fraction or probability of each mass eiginstates in a flavour state.

Question

What does wave packet in neutrino oscillation mean?

Answer

To Be Answered.

Question

How would a wave packet spread?

Answer

A Gaussian wave packet would spread or shrink. The key of this spreading or shrinking is the dispersion relation.

For **non-relativistic** Gaussian wave packet $\psi(x,t) = e^{-\alpha(k-k_0)^2}$ in momentum basis with dispersion relation $\hbar\omega = \frac{\hbar^2 k^2}{2m}$, the expansion of packet is

$$\Delta x = \sqrt{\alpha^2 + \left(\frac{\hbar t}{2m}\right)^2}.$$

Obviously, the RMS width spreads according to group velocity $v_g = \hbar_0/m$.

However, the situation could be different for a relativistic neutrino.

Question

What will scattering do to a wave packet.

Answer

Momentum transfer for a plan wave case in Born approximation is

---

Refs & Notes

CHAPTER 7

Vacuum Oscillation

Schrodinger equation is

$$i\partial_t |\Psi\rangle = \mathbf{H} |\Psi\rangle ,$$

where for relativistic neutrinos, the energy is

$$\mathbf{H}^{vm} = \begin{pmatrix} \sqrt{p^2+m_1^2} & 0 & 0 \\ 0 & \sqrt{p^2+m_2^2} & 0 \\ 0 & 0 & \sqrt{p^2+m_3^2} \end{pmatrix},$$

in which the energy terms are simplified using the relativistic condition

$$\begin{aligned} \sqrt{p^2+m_i^2} &= p\sqrt{1+\frac{m_i^2}{p^2}} \\ &\approx p(1+\frac{1}{2}\frac{m_i^2}{p^2}). \end{aligned}$$

So Called Decoherence

Here we assume that they all have the same energy but different mass. The thing is we assume they have the same velocity since the mass is very small. To have an idea of the velocity difference, we can calculate the distance travelled by another neutrino in the frame of one neutrino.

Assuming the mass of a neutrino is 1eV with energy 10MeV, we will get a speed of $1 - 10^{-14}$ c. This 10^{-14} c will make a difference about $3\mu\mathrm{m}$ in 1s.

Will decoherence happen due to this? For high energy neutrinos this won't be a problem however for low energy neutrinos this will definitely cause a problem for the wave function approach.Because the different mass eigenstates will become decoherent gradually along the path.

A estimation of the decoherence length is

$$l_{\mathrm{coh}} = \frac{v_g}{\Delta v_g}\sigma.$$

To obtain the relation,

$$\Delta x = |v_1 - v_2| t_{\mathrm{coh}}$$
$$\frac{\hbar c}{\Delta E} = |\frac{m_1^2}{2E_1^2} - \frac{m_2^2}{2E_2^2}| t_{\mathrm{coh}}$$
$$\frac{\hbar c}{\Delta E} = \frac{1}{2E} |\Delta m_{12}^2| t_{\mathrm{coh}}$$

It should be made clear that this is not really decoherence but in the view of wave packet formalism different propagation eigenstates will be far away from each other. As long as we put them together again we can overlap and oscillate again. No quantum decoherence is happening at all.

In general the flavor eigenstates are the mixing of the mass eigenstates with a unitary matrix $mathbf U$, that is

$$|\nu_\alpha\rangle = U_{\alpha i} |\nu_i\rangle ,$$

where the α s are indices for flavor states while the is are indices for mass eigenstates.

To find out the equation of motion for flavor states, plugin in the initary tranformation,

$$i U_{\alpha i} \partial_t |\nu_i\rangle = U_{\alpha i} H^m_{ij} |\nu_j\rangle .$$

We use index :math:{}^{vm}$ for representation of Hamiltonian in mass eigenstates in vacuum oscillations. Applying the unitary condition of the transformation,

$$\mathbf{I} = \mathbf{U}^\dagger \mathbf{U},$$

I get

$$i U_{\alpha i} \partial_t |\nu_i\rangle = U_{\alpha i} H^m_{ij} U^\dagger_{j\beta} U_{\beta k} |\nu_k\rangle ,$$

which is simplified to

$$i \partial_t |\nu_\alpha\rangle = H^f_{\alpha\beta} |\nu_\beta\rangle ,$$

since the transformation is time independent.

The new Hamiltonian in the representations of flavor eigenstates reads

$$H^f_{\alpha\beta} = U^\dagger_{\alpha i} H^m_{ij} U_{j\beta}.$$

Survival Problem

The neutrino states at any time can be written as

$$|\Psi(t)\rangle = X_1 |\nu_1\rangle e^{-iE_1 t} + X_2 |\nu_2\rangle e^{-iE_2 t},$$

where X_1 and X_2 are the initial conditions which are determined using the neutrino initial states.

Survival probalility is the squrare of the projection on an flavor eigenstate,

$$P_\alpha(t) = |\langle \nu_\alpha \mid \Psi(t)\rangle|^2.$$

The calculation of this expression requires our knowledge of the relation between mass eigenstates and flavor eigenstates which we have already found out.

Recall that the transformation between flavor and mass states is

$$\left|\nu_i\right\rangle = U^{-1}_{i\alpha}\left|\nu_\alpha\right\rangle,$$

which leads to the inner product of mass eigenstates and flavor eigenstates,

$$\begin{aligned}\langle\nu_\alpha \mid \nu_i\rangle &= \left\langle\nu_\alpha\right| U^{-1}_{i\beta}\left|\nu_\beta\right\rangle \\ &= U^{-1}_{i\beta}\delta_{\alpha\beta} \\ &= U^{-1}_{i\alpha}.\end{aligned}$$

The survival probability becomes

$$\begin{aligned}P_\alpha(t) &= |\langle\nu_\alpha \mid X_1\left|\nu_1\right\rangle e^{-iE_1 t}X_2\left|\nu_2\right\rangle e^{-iE_2 t}\rangle|^2 \\ &= |X_1 e^{-iE_1 t}\langle\nu_\alpha \mid |\nu_1\rangle\rangle + X_2 e^{-iE_2 t}\langle\nu_\alpha \mid \nu_2\rangle|^2 \\ &= |\sum_i X_i e^{-iE_i t}U^{-1}_{i\alpha}|^2 \\ &= \sum_i X_1^* e^{iE_i t}U^{\dagger *}_{i\alpha}\sum_i X_i e^{-iE_i t}U^{\dagger}_{i\alpha} \\ &= |X_1|^2 U^{\dagger *}_{1\alpha}U^{\dagger}_{1\alpha} + |X_2|^2 U^{\dagger *}_{2\alpha}U^{\dagger}_{2\alpha} + X_1^* X_2 U^{\dagger *}_{1\alpha}U^{\dagger}_{2\alpha}e^{iE_1 t - iE_2 t} + X_2^* X_1 U^{\dagger *}_{2\alpha}U^{\dagger}_{1\alpha}e^{iE_2 t - iE_1 t}\end{aligned}$$

$U^{\dagger *}_{i\alpha}$ stands for the \$i\$th row and the α th column of the matrix $U^{\dagger *}$.

Two Flavor States

Suppose the neutrinos are prepared in electron flavor initially, the survival probability of electron flavor neutrinos is calculated using the result I get previously.

Electron neutrinos are the lighter ones, then I have $_a = _e$ and denote $_b = _x$.

Meaning of Mixing

In the small mixing angle limit,

$$\begin{pmatrix}\nu_e \\ \nu_x\end{pmatrix} \to \begin{pmatrix}1 & \theta \\ -\theta & 1\end{pmatrix}\begin{pmatrix}\nu_1 \\ \nu_2\end{pmatrix}$$

which is very close to an identity matrix. This implies that electron neutrino is more like mass eigenstate ν_1 . By ν_1 we mean the state with energy $\frac{\delta m^2}{4E}$ in vacuum.

In fact the dynamics of the system is very easily solved without dive into the math. Suppose we have $\left|\nu_e\right\rangle$ initially, which is

$$\Psi(x=0) = \left|\nu_e\right\rangle = \cos\theta_v\left|\nu_1\right\rangle - \sin\theta_v\left|\nu_2\right\rangle,$$

the state of the system at distance x is directly written down

$$\begin{aligned}\Psi(x) &= \cos\theta_v\left|\nu_1\right\rangle e^{-iE_1 x} - \sin\theta_v\left|\nu_2\right\rangle e^{-iE_2 x} \\ &= e^{-iE_1 x}(\cos\theta_v\left|\nu_1\right\rangle - \sin\theta_v\left|\nu_2\right\rangle e^{i(E_1-E_2)x}).\end{aligned}$$

Since a global phase doesn't change the detection, we write the state as

$$\Psi(x) = \cos\theta_v\left|\nu_1\right\rangle - \sin\theta_v\left|\nu_2\right\rangle e^{i(E_1-E_2)x}.$$

Notice that the period of the expression is

$$l_v = \frac{2\pi}{E_1 - E_2} = -\frac{4\pi E}{\Delta m_{12}}.$$

Then the state becomes

$$\Psi(x) = \cos\theta_v \left|\nu_1\right\rangle - \sin\theta_v \left|\nu_2\right\rangle e^{i2\pi x/l_v}.$$

The survival probability for electron neutrinos is

$$\begin{aligned} P(\nu_e, L) &= 1 - \sin^2(2\theta_v)\sin^2\left(\frac{\Delta m^2 L}{4E}\right) \\ &= 1 - \frac{1}{2}\sin^2 2\theta_v\left(1 - \cos\left(\frac{2\pi x}{l_v}\right)\right) \end{aligned}$$

Refs and Notes

CHAPTER 8

Interaction With Matter

MSW Effect

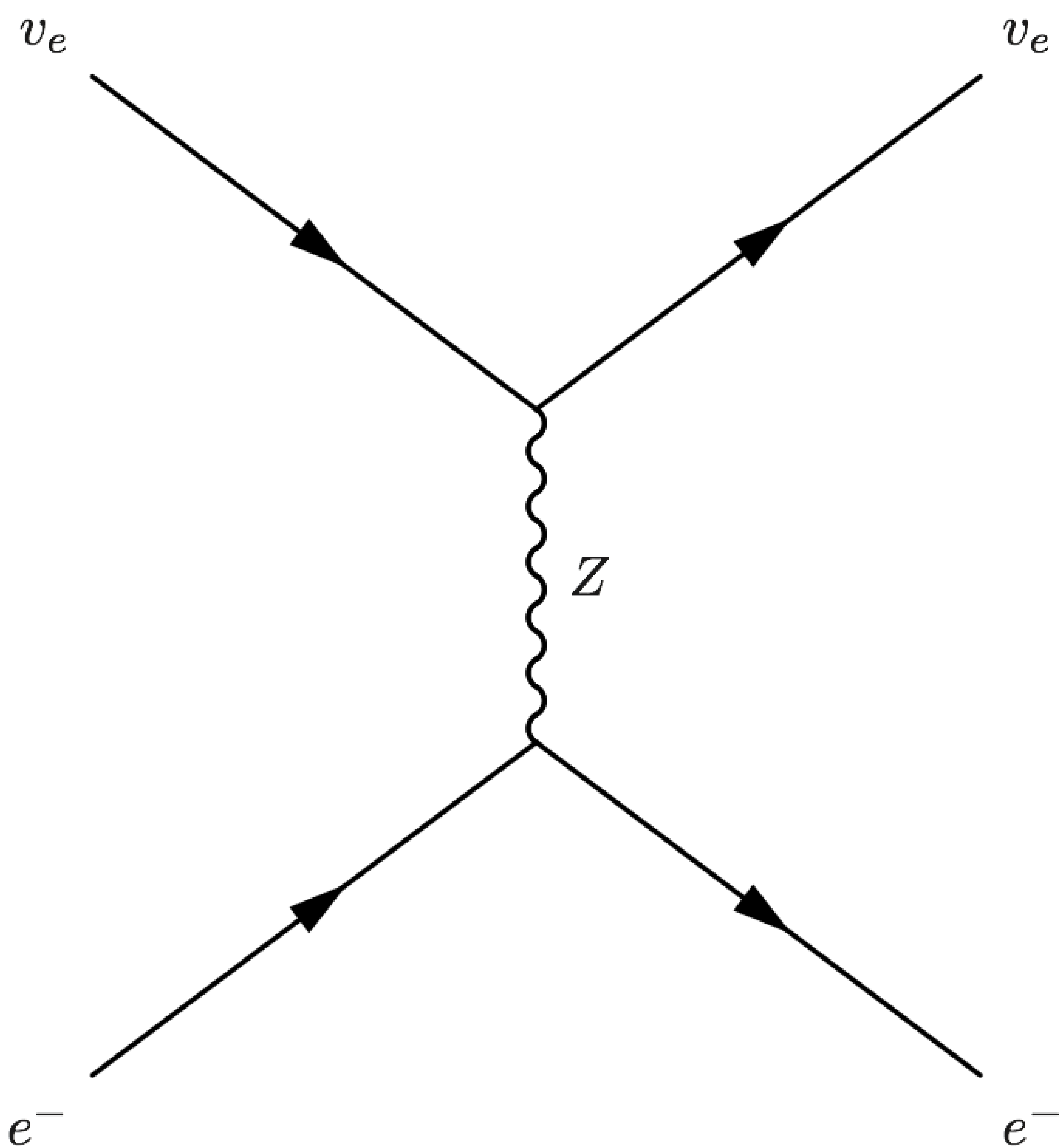
Physics of MSW

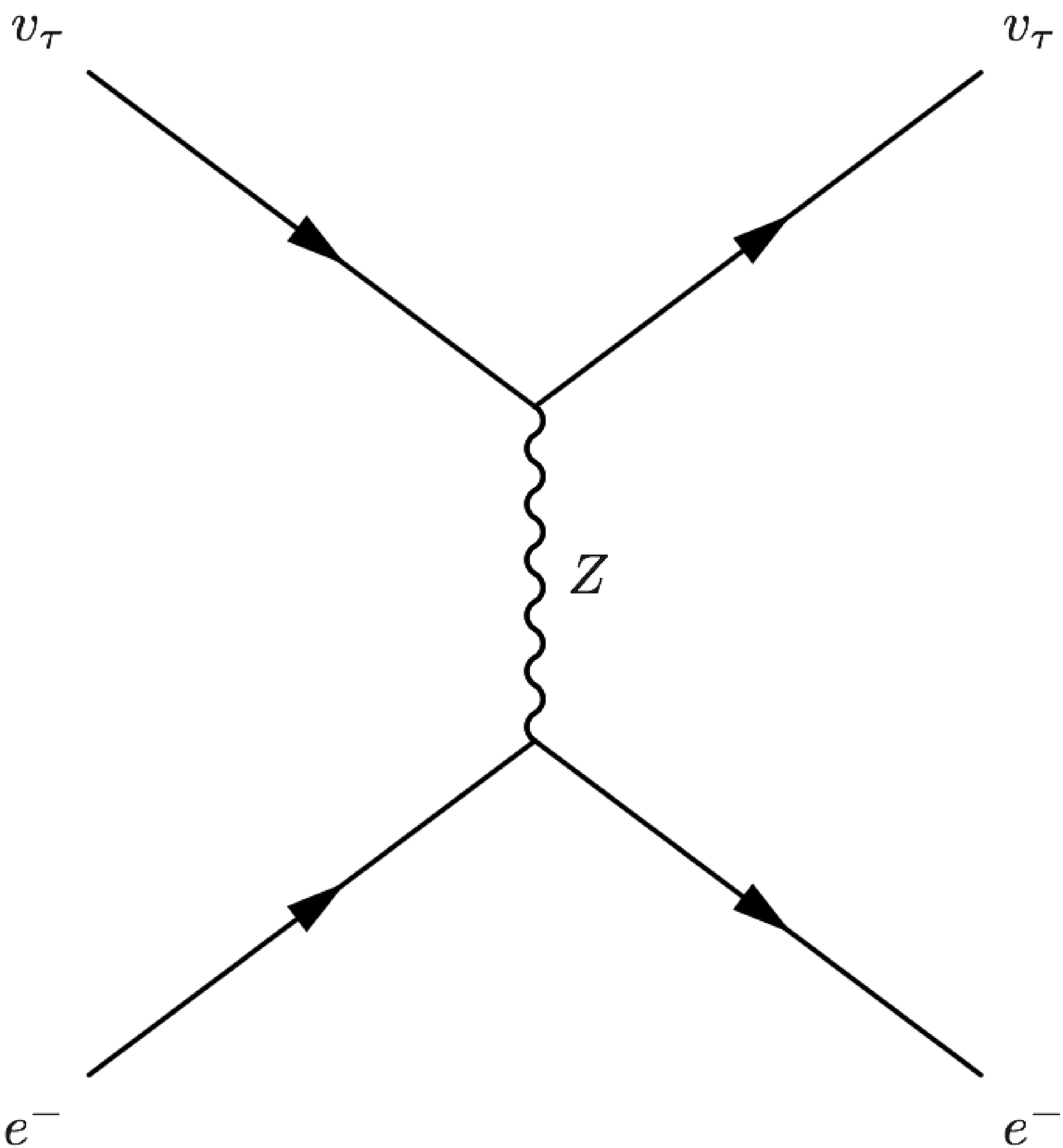
As neutrinos passing by matter, the effective mass coming from energy change becomes important thus changing it's eigenstates and propagation.

Neutrinos do interact with matter, mostly electrons in most cases.

```
\begin{fmfgraph*}(200,180)
  \fmfleft{i1,i2}
  \fmfright{o1,o2}
  \fmf{fermion}{i1,v1,o1}
  \fmf{fermion}{i2,v2,o2}
  \fmf{photon}{v1,v2}
  \fmflabel{$v_e$}{i2}
  \fmflabel{$e^-$}{i1}
  \fmflabel{$v_e$}{o2}
  \fmflabel{$e^-$}{o1}
  \fmf{photon,label=$Z$}{v1,v2}
\end{fmfgraph*}
```

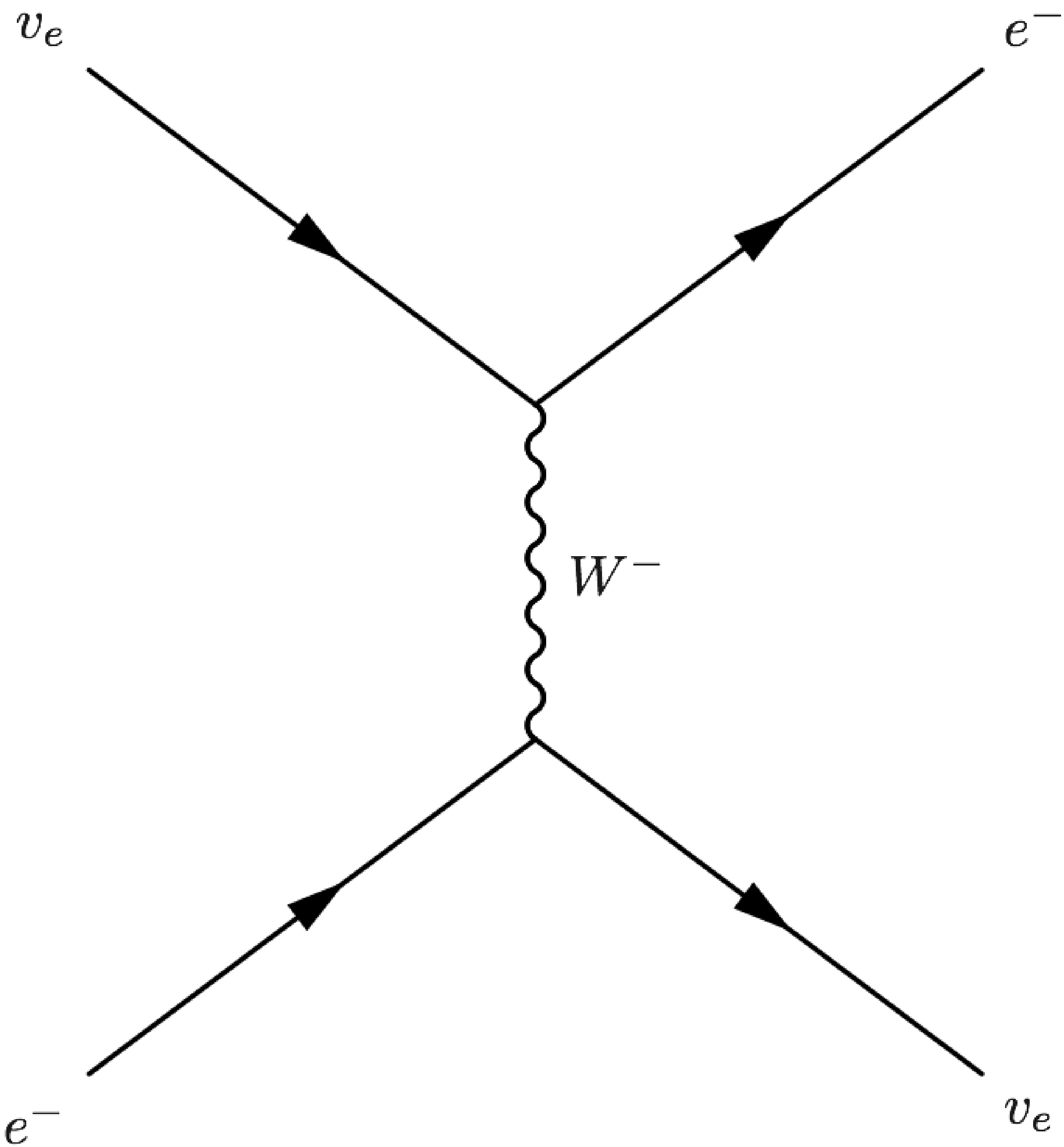
```
\begin{fmfgraph*}(200,180)
 \fmfleft{i1,i2}
 \fmfright{o1,o2}
 \fmf{fermion}{i1,v1,o1}
 \fmf{fermion}{i2,v2,o2}
 \fmf{photon}{v1,v2}
 \fmflabel{$v_\tau$}{i2}
 \fmflabel{$e^-$}{i1}
 \fmflabel{$v_\tau$}{o2}
 \fmflabel{$e^-$}{o1}
```


v_e
v_e
Z
e^-
e^-



v_τ
v_τ
Z
e^-
e^-

```
 \fmf{photon,label=$Z$}{v1,v2}
\end{fmfgraph*}
```




```
\begin{fmfgraph*}(200,180)
 \fmfleft{i1,i2}
 \fmfright{o1,o2}
 \fmf{fermion}{i1,v1,o1}
 \fmf{fermion}{i2,v2,o2}
 \fmf{photon}{v1,v2}
 \fmflabel{$v_e$}{i2}
```

```
 \fmflabel{$e^-$}{i1}
 \fmflabel{$v_e$}{o1}
 \fmflabel{$e^-$}{o2}
 \fmf{photon,label=$W^{-}$}{v1,v2}
\end{fmfgraph*}
```

The one that is missing is the charged current for nu_τ and e^- interaction because of lepton number conservation.

The first two diagrams will add two equal terms on the diagonal terms of Hamiltonian, which can be viewed as adding a number times identity matrix thus conserves the eigenstates while shifts the eigenvalues. However, the third diagram will only add a term to the first diagonal term of Hamiltonian, which is the weak coupling $\Delta = \sqrt{2} G_F n(x)$ with $n(x)$ being the number density of electrons.

Identity Matrix and Survival Probability

Identity matrix shifts the eigenvalues up and down homogeneously which changes the evolution of the state. However, since this is only a phase, the calculation of the survival probability will kill this phase.

Weak Interaction

We can guess this interaction term using physics picture. This interaction should be proportional to density of electrons with a coupling constant G_F. Then check the dimensions.

$$[G_F] = [E]^{-2}$$
$$[n(x)] = [E]^3$$

So the dimension is right. The missing constant is $\sqrt{2}$.

This symmetry breaking will change the evolution and makes the states more electron neutrino.

This is the reason of MSW effect.

In other words, the first requirement of MSW effect is that the electrons interacts with neutrinos and makes it in a specific state that is heavy if the electron density is strong enough. Meanwhile, if the mixing angle is not that large, a level crossing could happen making the state a light state as the density becomes vacuum. The other requirement, which is obvious, is that the density change should be adiabatic, the meaning of which is the density profile of matter gently reduces to vacuum, leaving enough reaction time for the neutrinos.

The MSW effect itself can be made clear using the example of neutrino oscillations in our sun.

Small Mixing Angle

Take two flavour mixing as an example.

$$\begin{pmatrix} \nu_e \\ \nu_x \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

In the small mixing angle limit,

$$\begin{pmatrix} \nu_e \\ \nu_x \end{pmatrix} \to \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

which is very close to an identity matrix. This implies that electron neutrino is more like mass eigenstate ν_1. By ν_1 we mean the state with energy $\frac{\delta m^2}{4E}$ in vacuum.

We need this intuitive picture to understand MSW effect. Electron neutrinos are almost identical to the low mass neutrino mass eigenstate. **However, as we will see, due to the matter interaction, the electron flavour neutrino is corresponding to the HEAVY mass eigenstate.** This is the key idea in physics of MSW effect.

The Hamiltonian for neutinos with neutrino-matter interaction (in flavour basis) is

$$\mathbf{H} = \frac{\delta m^2}{4E}\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} + \frac{\Delta}{2}\sigma_\mathbf{3} + \Delta \mathbf{I},$$

where the last term (green part) can be neglected because this term will only shift all the eigenvalues with the same amount without changing the eigenvectors.

Define a quantities like $\omega = \frac{\delta m^2}{2E}$ for neutrinos ($\bar\omega = \frac{\delta m^2}{-2E}$ for antineutrinos) and $\Delta = \sqrt{2} G_F n(x)$ (which might be denoted by $\nu = \sqrt{2} G_F n_\nu$ in other lituratures).

Using Pauli matrices, I can decompose this to

$$\mathbf{H} = \frac{\omega}{2} = (-\cos 2\theta\sigma_3 + \sin 2\theta\sigma_1) + \frac{\Delta}{2}\sigma_\mathbf{3} + \Delta \mathbf{I}$$

Note: As a reminder, $\Delta = \sqrt{2} G_F n(x)$.

Note: The red part is from the charged current Feynman diagram. We have a σ_3 matrix instead of an matrix like

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

because we rewrite this matrix with Pauli matrices and identy. Then the identities are neglected.

This can be done properly because Pauli matrice and Identy matrix form a complete basis.

In a more compact form, this Hamiltonian is

$$\begin{aligned}\mathbf{H} &= \frac{\delta m^2}{4E}(-\cos 2\theta\sigma_\mathbf{3} + \sin 2\theta\sigma_\mathbf{1}) + \frac{\Delta}{2}\sigma_\mathbf{3} \\ &= \left(\frac{\Delta}{2} - \frac{\delta m^2}{4E}\cos 2\theta\right)\sigma_\mathbf{3} + \frac{\delta m^2}{4E}\sin 2\theta\sigma_\mathbf{1}\end{aligned}$$

Note: Eigenvalues of $\sigma_\mathbf{3}$ are 1 and -1 with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

As we have mentioned, this Hamiltonian is in flavour basis. When mixing angle $\theta \to 0$, the eigenvectors are almost eigenvectors of $\sigma_\mathbf{3}$ which are electron neutrinos and x type neutrinos.

Interesting Limits

Before we really solve the equation of motion, some interesting limits can be shown here.

Interaction Δ is much larger than cacuum mixing terms. In this case, the Hamiltonian becomes diagonalized and the neutrinos will stay on it's flavour eigenstates in the propagation.

Interaction Δ is much smaller than vacuum mixing terms. The propagation reduces to vacuum case.

To see this effect quantitively, we need to diagonalize this Hamiltonian (**Can we actually diagonalize the equation of motion? NO!**). Equivalently, we can rewrite it in the basis of mass eigenstates $\{\lvert\nu_L(x)\rangle, \lvert\nu_H(x)\rangle\}$,

$$\lvert\nu_L(x)\rangle = \cos\theta(x)\lvert\nu_e\rangle - \sin\theta(x)\lvert\nu_\mu\rangle$$
$$\lvert\nu_H(x)\rangle = \sin\theta(x)\lvert\nu_e\rangle - \cos\theta(x)\lvert\nu_\mu\rangle .$$

This new rotation in matrix form is

$$\begin{aligned}\begin{pmatrix}\lvert\nu_L(x)\rangle \\ \lvert\nu_H(x)\rangle\end{pmatrix} &= \begin{pmatrix}\cos\theta(x) & -\sin\theta(x) \\ \sin\theta(x) & \cos\theta(x)\end{pmatrix}\begin{pmatrix}\lvert\nu_e\rangle \\ \lvert\nu_x\rangle\end{pmatrix} \\ &= \mathbf{U_x^{-1}}\begin{pmatrix}\lvert\nu_e\rangle \\ \lvert\nu_x\rangle\end{pmatrix}\end{aligned}$$

Diagonalize Hamiltonian

To diagonilize it, we need to multiply on both sides the rotation matrix and its inverse,

$$\mathbf{H_{xd}} = \mathbf{U_x^{-1}}\mathbf{H}\mathbf{U_x}.$$

The second step is to set the off diagonal elements to zero. By solving the equaions we can find the $\sin 2\theta(x)$ and $\cos 2\theta(x)$.

$$\begin{aligned}\mathbf{H_{xd}} &= \mathbf{U_x^{-1}}\left(A_1\sigma_\mathbf{1} + A_3\sigma_\mathbf{3}\right)\mathbf{U_x} \\ &= \begin{pmatrix}A_3\cos 2\theta(x) - A_1\sin 2\theta(x) & A_3\sin 2\theta(x) + A_1\cos 2\theta(x) \\ A_3\sin 2\theta(x) + A_1\cos 2\theta(x) & -A_3\cos 2\theta(x) + A_1\sin 2\theta(x)\end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}A_3 &= \frac{\Delta}{2} - \frac{\delta^2 m}{4E}\cos 2\theta \\ A_1 &= \frac{\delta^2 m}{4E}\sin 2\theta.\end{aligned}$$

Set the off-diagonal elements to zero,

$$A_3\sin 2\theta(x) + A_1\cos 2\theta(x) = 0$$

So the solutions are

$$\begin{aligned}\sin 2\theta(x) &= \frac{A_1}{\sqrt{A_1^2 + A_3^2}} \\ \cos 2\theta(x) &= \frac{-A_3}{\sqrt{A_1^2 + A_3^2}}.\end{aligned}$$

Plug in A_1 and A_3

$$\begin{aligned}\sin 2\theta(x) &= \frac{\sin 2\theta_v}{\sqrt{\left(\frac{\Delta}{\omega}\right)^2 + 1 - 2\frac{\Delta}{\omega}\cos 2\theta_v}} \\ \cos 2\theta(x) &= \frac{\cos 2\theta_v - \frac{\Delta}{\omega}}{\sqrt{\left(\frac{\Delta}{\omega}\right)^2 + 1 - 2\frac{\Delta}{\omega}\cos 2\theta_v}}.\end{aligned}$$

Define $\hat{\Delta} = \frac{\Delta}{\omega}$ with $\omega = \frac{\Delta m^2}{2E}$, which represents the matter interaction strength compared to the vacuum oscillation.

$$\sin 2\theta(x) = \frac{\sin 2\theta_v}{\sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta}\cos 2\theta_v}}$$
$$\cos 2\theta(x) = \frac{\cos 2\theta_v - \hat{\Delta}}{\sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta}\cos 2\theta_v}}.$$

This diagonalize the Hamiltonian LOCALLY. It's not possible to diagonalize the Hamiltonian globally if the electron number density is not a constant.

The point is, for equation of motion, we have a differentiation with respect to position x! So even we diagonalize the Hamiltonian, the equation of motion won't be diagonalized. An extra matrix will occur on the LHS and de-diagonalize the Hamiltonian on RHS.

Note: As $\Delta \to \infty$, $A_3 \to \infty$ and $\sin 2\theta(x)$ vanishes. Thus the neutrino will stay on flavour eigenstates.

With the newly defined heavy-light mass eigenstates, we can calculate the propagatioin of neutrinos,

$$i\hbar\partial_t \left|\psi_x(t)\right\rangle = \mathbf{ExtraMatrixFromLHS} \cdot \mathbf{H}_{xd} \left|\psi_x(t)\right\rangle,$$

where the $\mathbf{ExtraMatrixFromLHS}$ comes from the fact that changing from flavor basis $\Psi(x)$ to heavy-light basis $\Psi_m(x)$ using $\mathbf{U_m}$,

$$i\partial_x(\mathbf{U_m}\Psi_m(x)) = H(\mathbf{U_m}\Psi_m(x))$$

only returns

$$i\partial_x\Psi_m(x) = \mathbf{H_{md}}\Psi_m(x) - i\mathbf{U_m^{-1}}(\partial_x\mathbf{U_m})\Psi_m(x).$$

We imediately know the propagation is on the heavy-light mass eigenstates under adiabatic condition WITHOUT solving the equation. The eigenvalue of these states are $-\sqrt{A_3^2 + A_1^2}$ and $\sqrt{A_3^2 + A_1^2}$. The absolute value of these solutions grow as Δ becomes large.

Combining the two terms on RHS,

$$i\partial_x\Psi_m(x) = \mathbf{H_m}\Psi_m(x),$$

where

$$\mathbf{H_m} = \mathbf{H_{md}} - i\mathbf{U_m^{-1}}(\partial_x\mathbf{U_m}).$$

The only part inside $\mathbf{U_m(x)}$ that is space dependent is the number density of the electrons $n(x)$. **Thus we know immediately that the Hamiltonian is diagonalized if the number density is constant.**

Is Adabatic Condition Valid Here?

Haxton's paper.

Before going into the system, here is a discussion of adiabatic in thermodynamics.

From the two solutions we know there is a gap between the two trajectories. We draw a figure with electron number density as the horizontal axis and energy as the vertical axis.

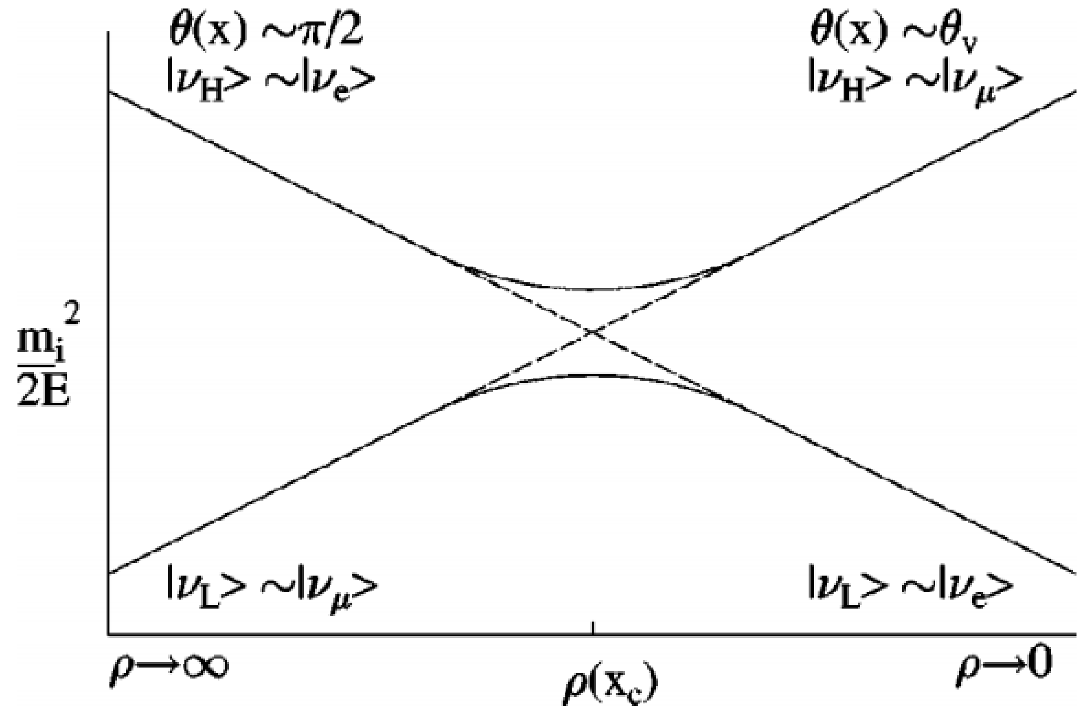


Fig. 9. Schematic illustration of the MSW crossing. The dashed lines correspond to the electron–electron and muon–muon diagonal elements of the m_ν^2 matrix in the flavor basis. Their intersection defines the level-crossing density ρ_c. The solid lines are the trajectories of the light and heavy local mass eigenstates. If the electron neutrino is produced at high density and propagates adiabatically, it will follow the heavy-mass trajectory, emerging from the sun as a ν_μ.

Fig. 8.1: Neutrino physics by Wick C. Haxton and Barry R. Holstein.

MSW Refraction, Resonance and More

Hysteresis Loops of Neutrino Oscillations Due to MSW Effect

Due to MSW effect, a system that is close to adiabaticity but not exactly adiabaticity could exhibit hysteresis effect, i.e., neutrinos going from high density region to low density region then coming back could form a hysteresis loop.

TODO

1. Write down the effective potential $V(x)$ which depends on the position. Refractive index is defined as $n_{ref}-1 = \frac{V}{p}$.
2. Two characteristic length: $l_v = \frac{4\pi E}{\delta m^2}$ as the vacuum oscillation length and $l_0 = \frac{2\pi}{V}$ as the refraction length. As the becomes comparable resonance occurs. For small mixing angle cases, resonance happens when vacuum length is about the length of refraction.

There are three different matrix representatioins that is useful to the calculations.

1. Flavor basis;
2. Vacuum mass eigenstate basis;
3. Instataneous mass eigenstate basis.

Basis of Hamiltonian

In vacuum mass eigenstate basis, the Hamiltonian without matter and self-interaction is easy and straightforward,

$$\mathbf{H_{vmv}} = \mathbf{H_{vp}} = \frac{1}{2E}\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix}.$$

To remove the trace, we can subtract a identity matrix

$$\begin{aligned} &\mathbf{H} - \frac{m_1^2}{2E}\mathbf{I} \\ =&\frac{1}{2E}\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} - \frac{m_1^2}{2E}\mathbf{I} \\ =&\frac{1}{2E}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta m_{12}^2 & 0 \\ 0 & 0 & \Delta m_{13}^2 \end{pmatrix} \end{aligned}$$

The interaction in flavor basis is

$$\mathbf{V_f} = \begin{pmatrix} \sqrt{2}G_F n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To write down the Hamiltonian in vacuum mass eigenstates, we transform the interaction term to vacuum mass eigenstates by

$$\mathbf{V_{vm}} = \mathbf{U^{-1}VU},$$

where U is the PMNS matrix.

To write down the Hamiltonian in flavor basis, we transform the vacuum Hamiltonian to flavor basis **after remove the trace**, which is

$$\mathbf{H_{fv}} = \mathbf{U}\mathbf{H_{vmv}}\mathbf{U^{-1}}.$$

We could also write down the Hamiltonian matrix in instantaneous mass eigenstates, which requires a instantaneous diagonalization.

2 Flavor Neutrino Oscillations and Matter Effect

Solar Neutrinos

Electron neutrinos are produced in the core of the sun then the neutrinos would propagate out to the surface of the sun without much difficulty. What is the predicted neutrino survival probability?

Interaction with matter plays a big role in neutrino oscillation. As shown previously, the interaction only affects (anti) electron neutrinos. In other words, the interaction term in flavor basis is

$$V_f = \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}.$$

where $\Delta = \sqrt{2} G_F n$ and n is the number density of the electrons. However, to do calculations, since identity matrix doesn't change the survival probability, we can always make the hamiltonian traceless, which becomes

$$H_i = \frac{\Delta}{2}\boldsymbol{\sigma_3}.$$

Constant Electron Number Density

Suppose we have an environment with constant electron number density, the term $-i\mathbf{U_m^{-1}}(\partial_x \mathbf{U_m})$ goes away. All we have is the diagonalized new Hamiltonian $\mathbf{H_{md}}$ and the eigenvalues are easily obtained which are

$$\begin{aligned} E_1 &= A_3 \cos 2\theta(x) - A_1 \sin 2\theta(x) \\ E_2 &= -A_3 \cos 2\theta(x) + A_1 \sin 2\theta(x). \end{aligned}$$

The final result for these two eigenvalues are

$$\begin{aligned} E_1 &= -\sqrt{\frac{\Delta^2 + \omega^2}{4} - \frac{\Delta\omega}{2}\cos 2\theta_v}. \\ E_2 &= \sqrt{\frac{\Delta^2 + \omega^2}{4} - \frac{\Delta\omega}{2}\cos 2\theta_v}.. \end{aligned}$$

Meanwhile the eigenstates are denoted as $|\nu_{c1}\rangle$ '$and : math$:ket{nu_{c2}}'.

Two Special Cases

Two special cases,

1. $\cos 2\theta_v \to 0$;
2. $\cos 2\theta_v \to 1$.

As for the survival probability for the initial condition that $\Psi(x=0) = |\nu_{c1}\rangle$, the result has the same form as the vacuum case, which is

$$P_x(\nu_e, L) = 1 - \sin^2(2\theta_m)\sin^2\left(\frac{\omega_m L}{2}\right),$$

where

$$\sin 2\theta(x) = \frac{\omega \sin 2\theta_v}{\sqrt{\omega^2 + \Delta^2 - 2\omega\Delta\cos 2\theta_v}}.$$

$\theta_m = \theta(x)$ is the effective mixing angle which in fact doesn't depend on x if the matter profile is constant.

Vacuum Survival Probability

As an comparison, the vacuum result is

$$P_x(\nu_e, L) = 1 - \sin^2(2\theta)\sin^2\left(\frac{\omega L}{2}\right),$$

for all electron flavor initial condition.

Adiabatic Limit

In some astrophysical environments the electron number density changes very slowly which means the term $\mathbf{U_m^{-1}}\partial_x \mathbf{U_m}$ is much smaller than $\mathbf{H_{md}}$. By intuition we would expect that this term could be dropped to the lowest order.

The eigen energies are slowing changing with the position of neutrinos,

$$E_1 = -\frac{\omega}{2}\sqrt{\hat\Delta^2 + 1 - 2\hat\Delta\cos 2\theta_v}$$
$$E_2 = \frac{\omega}{2}\sqrt{\hat\Delta^2 + 1 - 2\hat\Delta\cos 2\theta_v}.$$

When the term $\hat\Delta$ is very small $1 - 2\hat\Delta\cos 2\theta_v$ will dominate and the whole term decreases. On the other hand as $\hat\Delta$ becomes large, $\hat\Delta^2$ will dominate and the whole term grows. Mathematically we could find the region when the part $\sqrt{\hat\Delta^2 + 1 - 2\hat\Delta\cos 2\theta_v}$ decreases and increases.

The survival probability for the light neutrinos would be

$$P_x(\nu_L, L) = 1 - \sin^2(2\theta(x))\sin^2\left(\frac{\omega L}{2}\right).$$

The survival probability for electron flavor neutrino is

$$P_x(\nu_e, L) = \frac{1}{2} + \frac{1}{2}\cos 2\theta(x_0)\cos 2\theta_v,$$

if the neutrinos are produced in dense region and the detection happens in vacuum.

Adiabatic Limit of Nuetrino Oscillations in Matter

Before we move on to higher order corrections, it would be nice to understand this phenomenon.

1. The vacuum oscillation length can be extracted from vacuum oscillation survival probability. It is $L_v = \frac{2\pi}{\omega}$.

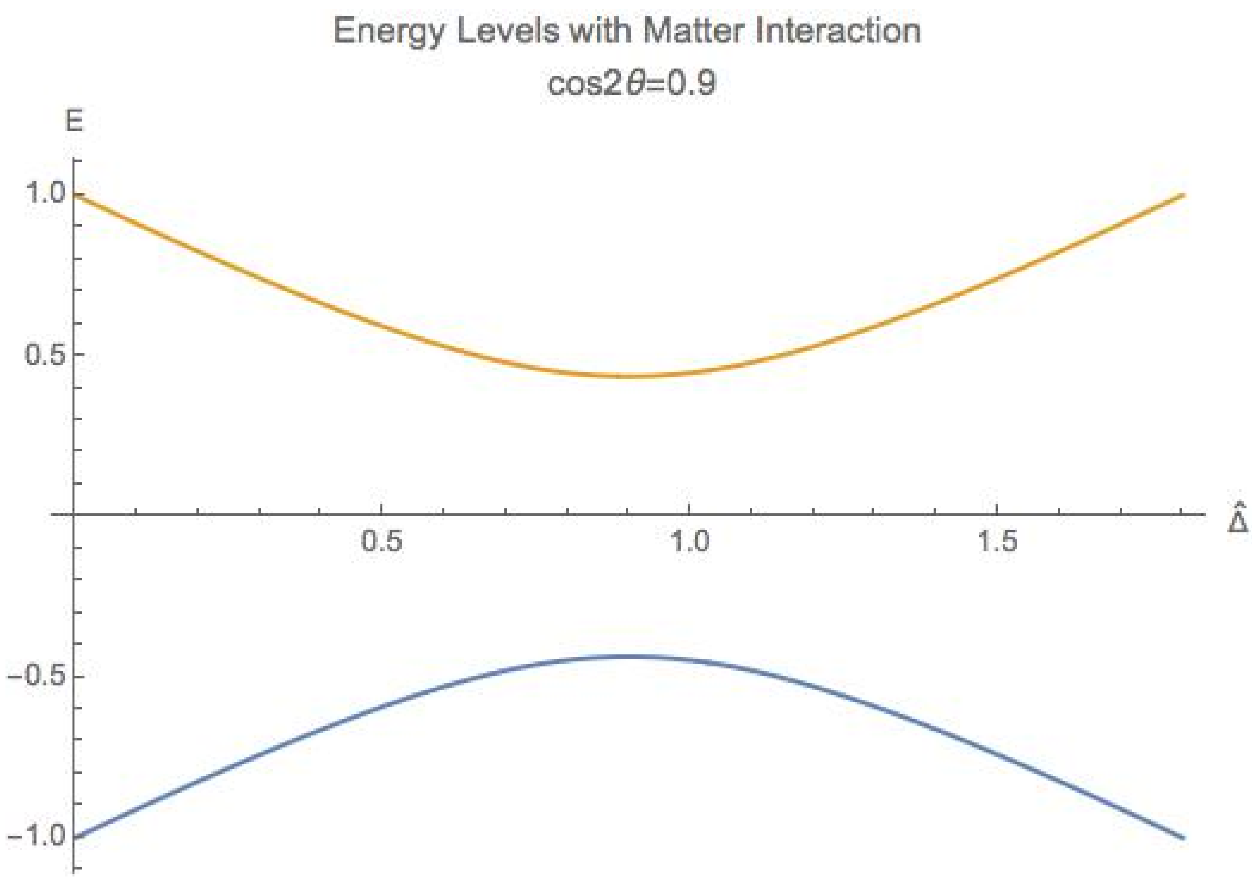


Fig. 8.2: Energy Levels for MSW effect. We have the up-down symmetry since we shifted the energy by a constant to remove the identity matrix in the Hamiltonian.

2. In this problem we have another energy scale which is the interaction, Δ. Here we can define another characteristic length $l_m = \frac{2\pi}{\Delta}$.
3. MSW resonance happens when the two character lengths are matching with each other. Another way to put it is that the term $\sin 2\theta(x)$ is minimized so that we have the smallest energy gap which leads to $\hat{\Delta} = \cos 2\theta_v$. Equivalently this is the relation

$$l_0 = l_m \cos 2\theta_v.$$

4. At resonance, we have

$$\cos 2\theta(x) = 1$$
$$\sin 2\theta(x) = 0.$$

   This is max mixing of the states which means that at the resonance point

$$\begin{pmatrix} \nu_L(x_r) \\ \nu_H(x_r) \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_x \end{pmatrix}$$

5. Resonance conditions corresponds to a resonance density which is given by

$$n_e(x) = \frac{\omega}{\sqrt{2}G_F} \cos 2\theta_v \equiv n_0(E, \Delta m^2) \cos 2\theta_v,$$

   where $n_0(E, \Delta m^2) = \frac{\omega}{\sqrt{2}G_F}$ is a characteristic number density which depends on the energy mixing angles and Δm^2 of the neutrinos.
6. One should notice that if the condition $\sin^2 2\theta(x) = \sin^2 2\theta_v$ is satisfied, the survival probability for $|\nu_1\rangle$ has the same **the form of** vacuum oscillation survival probability for electron neutrinos. The condition is solved,

$$\hat{\Delta}^2 + 1 - 2\hat{\Delta} \cos 2\theta_v = 1,$$

   which leads to

$$\hat{\Delta} = 0 \quad \text{or} \quad 2\cos 2\theta_v.$$

   The first condition is trivial which corresponds to vacuum however the second condition $Delta = 2cos 2theta_v omega$ means the interaction oscillation length is doubled compared to resonance point.

   Nevertheless, we should always remember to check what survival probability the expression is describing. Here we have survival probability for :math:'nu_L(x)'. At $n(x) \to 0$ the oscillation becomes vacuum oscillation.

General Discussion of Matter Effect

This part is a very general discussion of the matter effect *[Parke1986]*.

To work in flavor basis, we use the subscript $_{mf}$ to denote the flavor basis representation with mass effect. The equation of motion in flavor basis can be written down as

$$i\partial_x \Psi_{mf}(x) = \mathbf{H_{mf}} \Psi_{mf}(x)$$

where

$$\mathbf{H_{mf}} = \left(\frac{\Delta}{2} - \frac{\omega}{2} \cos 2\theta_v \right) \boldsymbol{\sigma_3} + \frac{\omega}{2} \sin 2\theta_v \boldsymbol{\sigma_1}.$$

There are three stages for neutrinos to travel from the core of the sun to vacuum.

1. At the core, electron neutrinos are produced. The electron flavor state should be projected onto heavy and light instantaneous mass eigenstates. What fallows is the that the propagation is adiabatic until the transition happens. As we have seen in adiabatic situation, the states will stay in heavy and light states all along the evolution if the system starts from heavy or light state,

$$\begin{aligned} \lvert \nu_{a1}(x) \rangle &= \exp(-i \int_0^x \frac{\omega_m(x)}{2} dx) \lvert \nu_L(x) \rangle \\ \lvert \nu_{a2}(x) \rangle &= \exp(i \int_0^x \frac{\omega_m(x)}{2} dx) \lvert \nu_H(x) \rangle , \end{aligned}$$

where the heavy and light states are defined in the adiabatic situation previously. **This is what happens before the passing through of the resonance.**

2. At the resonance point, light instantaneous mass eigenstate has a probability to jump to the heavy state and vice versa. When it comes to the resonance point which is non-adiabatic propagation, the transition between the states $\lvert \nu_L \rangle \to a_L \lvert \nu_L(x) \rangle + a_H \lvert \nu_H(x) \rangle \lvert \nu_1(x) \rangle$ and $\lvert \nu_H \rangle \to b_L \lvert \nu_L(x) \rangle + b_H \lvert \nu_H(x) \rangle$ will mix the heavy and light state up.

$$\begin{aligned} \lvert \nu_1(x) \rangle &= a_L \exp(-i \int_{x_r}^x \omega_m(x')/2 dx') \lvert \nu_L(x) \rangle + a_H \exp(i \int_{x_r}^x \omega_m(x')/2 dx') \lvert \nu_H(x) \rangle \\ \lvert \nu_2(x) \rangle &= b_L \exp(-i \int_{x_r}^x \omega_m(x')/2 dx') \lvert \nu_L(x) \rangle + b_H \exp(i \int_{x_r}^x \omega_m(x')/2 dx') \lvert \nu_H(x) \rangle , \end{aligned}$$

where the relations between the constants are determined using the condition that $\lvert \nu_1(x) \rangle$ and $\lvert \nu_2(x) \rangle$ are orthonormal, which leads to the conclusion that

$$\begin{aligned} b_L &= -a_H^* \\ b_H &= a_L^* \\ \lvert a_L \rvert^2 &= -\lvert a_H \rvert^2 . \end{aligned}$$

3. After the resonance point, the heavy and light states will continue on their adiabatic propagation.

Helpful Notes

The relation between θ_m and θ_v is given by

$$\omega_m \sin 2\theta_m = \omega \sin 2\theta_v .$$

Electron neutrinos are produced in a dense region as $\lvert \nu_e \rangle$, which are partially transformed to other the other neutrinos due to matter and the resonance then it propagates as if it satisfies the adiabatic condition again. The initial state in terms of light and heavy state is

$$\lvert \Psi_m(x_0) \rangle = \lvert \nu_e \rangle = \cos\theta_m(x_0) \lvert \nu_L(x_0) \rangle + \sin\theta_m(x_0) \lvert \nu_H(x_0) \rangle .$$

The final state right before the resonance is

$$\lvert \Psi_m(x_{r-}) \rangle = \cos\theta_m(x_0) \exp\left(-i \int_{x_0}^{x_{r-}} \frac{\omega_m(x)}{2} dx \right) \lvert \nu_L(x_{r-}) \rangle + \sin\theta_m(x_0) \exp\left(i \int_{x_0}^{x_{r-}} \frac{\omega_m(x)}{2} dx \right) \lvert \nu_H(x_{r-}) \rangle$$

After the resonance the state is described by the general jumping

$$\begin{aligned} \lvert \Psi_m(x) \rangle = &\cos\theta_m(x_0) \exp\left(-i \int_{x_0}^{x_{r-}} \frac{\omega_m(x)}{2} dx \right) \left(a_L \exp(-i \int_{x_r}^x \frac{\omega_m(x')}{2} dx') \lvert \nu_L(x) \rangle + a_H \exp(i \int_{x_r}^x \frac{\omega_m(x')}{2} dx') \lvert \nu_H(x) \rangle \right) \\ &+ \sin\theta_m(x_0) \exp\left(i \int_{x_0}^{x_{r-}} \frac{\omega_m(x)}{2} dx \right) \left(-a_H^* \exp(-i \int_{x_r}^x \frac{\omega_m(x')}{2} dx') \lvert \nu_L(x) \rangle + a_L^* \exp(i \int_{x_r}^x \frac{\omega_m(x')}{2} dx') \lvert \nu_H(x) \rangle \right) \end{aligned}$$

in which the x_{r-} is actually x_r thus

$$
\begin{aligned}
|\Psi_m(x)\rangle =& \cos\theta_m(x_0) \exp\left(-i\int_{x_0}^{x_r} \frac{\omega_m(x)}{2} dx \right) \left(a_L \exp(-i\int_{x_r}^{x} \frac{\omega_m(x')}{2} dx') \left|\nu_L(x)\right\rangle + a_H \exp(i\int_{x_r}^{x} \frac{\omega_m(x')}{2} dx') \left|\nu_H(x)\right\rangle \right) \\
&+ \sin\theta_m(x_0) \exp\left(i\int_{x_0}^{x_{r-}} \frac{\omega_m(x)}{2} dx \right) \left(-a_H^* \exp(-i\int_{x_r}^{x} \frac{\omega_m(x')}{2} dx') \left|\nu_L(x)\right\rangle + a_L^* \exp(i\int_{x_r}^{x} \frac{\omega_m(x')}{2} dx') \left|\nu_H(x)\right\rangle \right)
\end{aligned}
$$

To calculate the survival probability it is easier to use flavor basis, hence we have another form of $|\Psi_m(x)\rangle$ which is

$$
\begin{aligned}
|\Psi_m(x)\rangle =& \Big[\cos\theta_m(x_0) \exp\left(-i\int_{x_0}^{x_r} \frac{\omega_m(x')}{2} dx' \right) a_L \exp(-i\int_{x_r}^{x} \frac{\omega_m(x')}{2} dx') \\
& - \sin\theta_m(x_0) \exp\left(i\int_{x_0}^{x_{r-}} \frac{\omega_m(x')}{2} dx' \right) a_H^* \exp(-i\int_{x_r}^{x} \frac{\omega_m(x')}{2} dx') \Big] \left|\nu_L(x)\right\rangle \\
& + \Big[\cos\theta_m(x_0) \exp\left(-i\int_{x_0}^{x_r} \frac{\omega_m(x)}{2} dx \right) a_H \exp(i\int_{x_r}^{x} \frac{\omega_m(x')}{2} dx') \\
& + \sin\theta_m(x_0) \exp\left(i\int_{x_0}^{x_{r-}} \frac{\omega_m(x)}{2} dx \right) a_L^* \exp(i\int_{x_r}^{x} \frac{\omega_m(x')}{2} dx') \Big] \left|\nu_H(x)\right\rangle \\
=& \Big[\cos\theta_m(x_0) \exp\left(-i\int_{x_0}^{x_r} \frac{\omega_m(x)}{2} dx \right) a_L \exp(-i\int_{x_r}^{x} \frac{\omega_m(x')}{2} dx') \\
& - \sin\theta_m(x_0) \exp\left(i\int_{x_0}^{x_{r-}} \frac{\omega_m(x)}{2} dx \right) a_H^* \exp(-i\int_{x_r}^{x} \frac{\omega_m(x')}{2} dx') \Big] \left(\cos\theta_m(x) \left|\nu_e\right\rangle - \sin\theta_m(x) \left|\nu_x\right\rangle \right) \\
& + \Big[\cos\theta_m(x_0) \exp\left(-i\int_{x_0}^{x_r} \frac{\omega_m(x)}{2} dx \right) a_H \exp(i\int_{x_r}^{x} \frac{\omega_m(x')}{2} dx') \\
& + \sin\theta_m(x_0) \exp\left(i\int_{x_0}^{x_{r-}} \frac{\omega_m(x)}{2} dx \right) a_L^* \exp(i\int_{x_r}^{x} \frac{\omega_m(x')}{2} dx') \Big] \left(\sin\theta_m(x) \left|\nu_e\right\rangle + \cos\theta_m(x) \left|\nu_x\right\rangle \right)
\end{aligned}
$$

Since $\cos\theta_m$, $\sin\theta_m$ and ω_m are real while a_L and a_H are complex, survival amplitude of electron neutrinos is given by

$$
\begin{aligned}
& \langle \Psi_m(0) \mid \Psi_m(x) \rangle \\
=& \langle \nu_e \mid \Psi_m(x) \rangle \\
=& \Big[\cos\theta_m(x_0) \exp\left(-i\int_{x_0}^{x_r} \frac{\omega_m(x')}{2} dx' \right) a_L \exp(-i\int_{x_r}^{x} \frac{\omega_m(x')}{2} dx') \\
& - \sin\theta_m(x_0) \exp\left(i\int_{x_0}^{x_r} \frac{\omega_m(x')}{2} dx' \right) a_H^* \exp(-i\int_{x_r}^{x} \frac{\omega_m(x')}{2} dx') \Big] \cos\theta_m(x) \\
& + \Big[\cos\theta_m(x_0) \exp\left(-i\int_{x_0}^{x_r} \frac{\omega_m(x')}{2} dx' \right) a_H \exp(i\int_{x_r}^{x} \frac{\omega_m(x')}{2} dx') \\
& + \sin\theta_m(x_0) \exp\left(i\int_{x_0}^{x_r} \frac{\omega_m(x')}{2} dx' \right) a_L^* \exp(i\int_{x_r}^{x} \frac{\omega_m(x')}{2} dx') \Big] \sin\theta_m(x) \\
=& A_L \exp\left(-i\int_{x_r}^{x} \frac{\omega_m(x')}{2} dx' \right) + A_H \exp\left(i\int_{x_r}^{x} \frac{\omega_m(x')}{2} dx' \right),
\end{aligned}
$$

where the coefficients are

$$
\begin{aligned}
A_L(x) &= \cos\theta_m(x) \left[a_L \cos\theta_m(x_0) \exp\left(-i\int_{x_0}^{x_r} \frac{\omega_m(x')}{2} dx' \right) - a_H^* \sin\theta_m(x_0) \exp\left(i\int_{x_0}^{x_r} \frac{\omega_m(x')}{2} dx' \right) \right] \\
A_H(x) &= \sin\theta_m(x) \left[a_H \cos\theta_m(x_0) \exp\left(-i\int_{x_0}^{x_r} \frac{\omega_m(x')}{2} dx' \right) + a_L^* \sin\theta_m(x_0) \exp\left(i\int_{x_0}^{x_r} \frac{\omega_m(x')}{2} dx' \right) \right].
\end{aligned}
$$

The detection is in a region where matter density is very small, thus we use $x \to \infty$ which means the effective mixing angle becomes vacuum mixing angle. The probability is the square of the amplitude,

$$\begin{aligned}P(\nu_e, x) &= |\langle \Psi_m(0) \mid \Psi_m(x)\rangle|^2 \\ &= |A_L(x) \exp\left(-i \int_{x_r}^{x} \frac{\omega_m(x')}{2} dx'\right) + A_H(x) \exp\left(i \int_{x_r}^{x} \frac{\omega_m(x')}{2} dx'\right)|^2 \\ &= |A_L(x)|^2 + |A_H(x)|^2 + A_L^*(x) A_H(x) \exp(2i\phi) + A_H^*(x) A_L(x) \exp(-2i\phi) \\ &= |A_L(x)|^2 + |A_H(x)|^2 + 2\mathbf{Re}(A_L^*(x) A_H(x) \exp(2i\phi)),\end{aligned}$$

where ϕ is defined as

$$\phi = \int_{x_r}^{x} \frac{\omega_m(x')}{2} dx'.$$

Note that for any complex number $(a + ib)e^{i\phi} \equiv \rho e^{i\psi}$,

$$(a + ib)e^{i\phi} + c.c. = 2\rho \cos(\psi + \phi),$$

which means that the previous result can be simplified to

$$\begin{aligned}P(\nu_e, x) &= |A_L(x)|^2 + |A_H(x)|^2 + 2\mathbf{Re}(A_L^*(x) A_H(x) \exp(2i\phi)) \\ &= |A_L(x)|^2 + |A_H(x)|^2 + 2|A_L^*(x) A_H(x)| \cos\left(2\phi + \psi_{LH}\right),\end{aligned}$$

with the definition that $\psi_{LH}(x)$ is the argument of $A_L^*(x)A_H(x)$.

However the coefficients a_L and a_H are still not known yet. The trick is to average over the detection and production position. The average over x removes the cos term due to the dependent of x for ϕ and averages $\cos^2\theta_m(x)$ to $\frac{1}{2}$, which results in

$$\begin{aligned}\langle P(\nu_e, x)\rangle_x =& \cos^2\theta_m(x)(|a_H|^2 \cos^2\theta_m(x_0) + |a_L|^2 \sin^2\theta_m(x_0)) \\ &+ \sin^2\theta_m(x)(|a_H|^2 \cos^2\theta_m(x_0) + |a_L|^2 \sin^2\theta_m(x_0)) \\ &+ (-\cos^2\theta_m(x) + \sin^2\theta_m(x)) \cos\theta_m(x_0) \sin\theta_m(x_0)(a_H a_L e^{-2i\phi'} + \text{c.c}).\end{aligned}$$

Applying the condition that $|a_L|^2 + |a_H|^2 = 1$, the probability becomes

$$\langle P(\nu_e, x)\rangle_x = \frac{1}{2} + \frac{1}{2}(1 - 2|a_H|^2) \cos 2\theta_m(x_0) \cos 2\theta_v - |a_H a_L| \sin 2\theta_m(x_0) \cos 2\theta_v \cos(2\phi' + \psi_{LH}),$$

where ψ_{LH} is the argument of $a_H a_L$ and ϕ is $\int_{x_0}^{x_r} \frac{\omega_m(x')}{2} dx'$.

The average over production removes the last part.

Notice that in fact the detection happens in vacuum, which means $\theta_m(x) = \theta_v$.

$$\langle\langle P(\nu_e, x)\rangle_x\rangle_{x_0} = \frac{1}{2} + \frac{1}{2}(1 - 2|a_H|^2) \cos 2\theta_m(x_0) \cos 2\theta_v.$$

This means that the adiabatic result is of the form

$$P(\nu_e, x)_{\text{adiabatic}} = \frac{1}{2}(1 + \cos 2\theta_m \cos 2\theta_v).$$

Define a transition probability at resonance

$$P_r(\nu_L \to \nu_H) = |a_2|^2,$$

which can be determined by the Landau-Zener transition analytically (first order) to the first order.

Refs and Notes

1. Wolfenstein, L. (1978). Neutrino oscillations in matter. Physical Review D, 17(9), 2369–2374. doi:10.1103/PhysRevD.17.2369
2. Wolfenstein, L. (1979). Neutrino oscillations and stellar collapse. Physical Review D, 20(10), 2634–2635. doi:10.1103/PhysRevD.20.2634
3. Parke, S. J. (1986). Nonadiabatic Level Crossing in Resonant Neutrino Oscillations. Physical Review Letters, 57(10), 1275–1278. doi:10.1103/PhysRevLett.57.1275
4. Bethe, H. A. (1986). Possible Explanation of the Solar-Neutrino Puzzle. Physical Review Letters, 56(12), 1305–1308. doi:10.1103/PhysRevLett.56.1305

CHAPTER 9

Collective Behavior

In a dense neutrino environment, neutrino oscillations could exhibit collective behaviors or synchronized behaviors.

The key of such a behavior is the self interaction between neutrinos.

Phonon

In solid state physics, phonons are the collective behavior of atom or molecule oscillations. The necessary condition for such a behavior is the interaction between atoms or molecules.

Backgrounds of collective effect:

1. Matter background
2. Neutrino background a) sychronized oscillations: neutrino neutrino interaction potential is large compared to-ordinary oscillation frequencies in vacuum/medium + large asymmetry between neutrino and antineutrino distributions b) bipolar oscillations: neutrino and antineutrino oscillate in opposite directions; non-zero vacuum mixing angle + some conditions of mass hierarchy. neutrino-neutrino interaction ($\mu = \sqrt{2} G_F n_\nu$) is larger than vacuum oscillatioin frequency $\omega = \Delta m$. like a torque applys to a top where instabilities happpen as the torque force is too big (top wobbles and flips).

Ref

1. Raffelt, G. & Smirnov, A. Self-induced spectral splits in supernova neutrino fluxes. *Phys. Rev. D* **76**, (2007). (This paper includes a very brief summary of sychronized and bipolar.)

Collective Phenomenon

Neutrino-neutrino interaction can be described by the following Feymann diagram.

Electron neutrinos can exchange momentum with other neutrinos including itself. Suppose we have a muon neutrino moving forward, and vacuum oscillations,

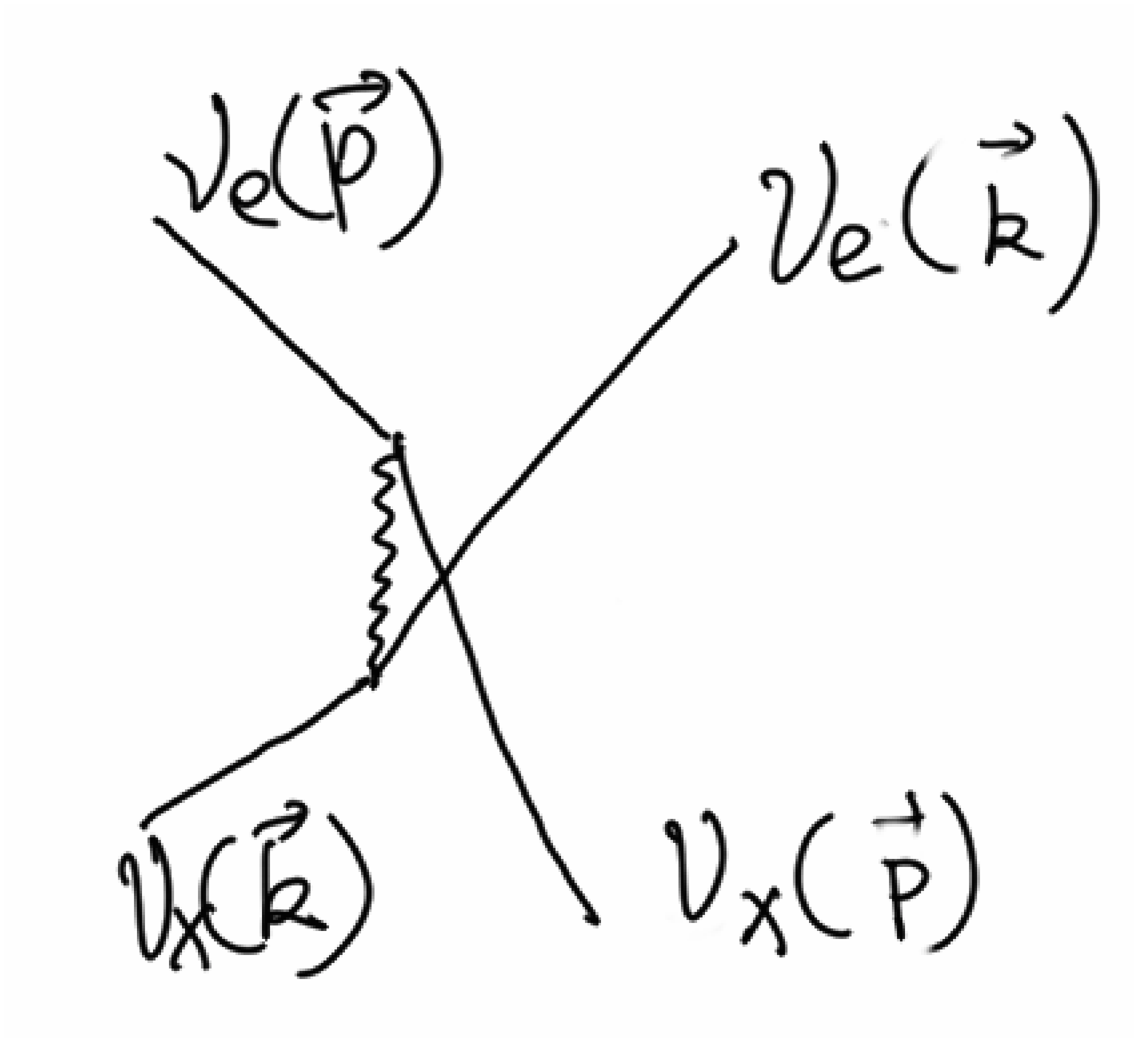


Fig. 9.1: They just exchange their momenta.

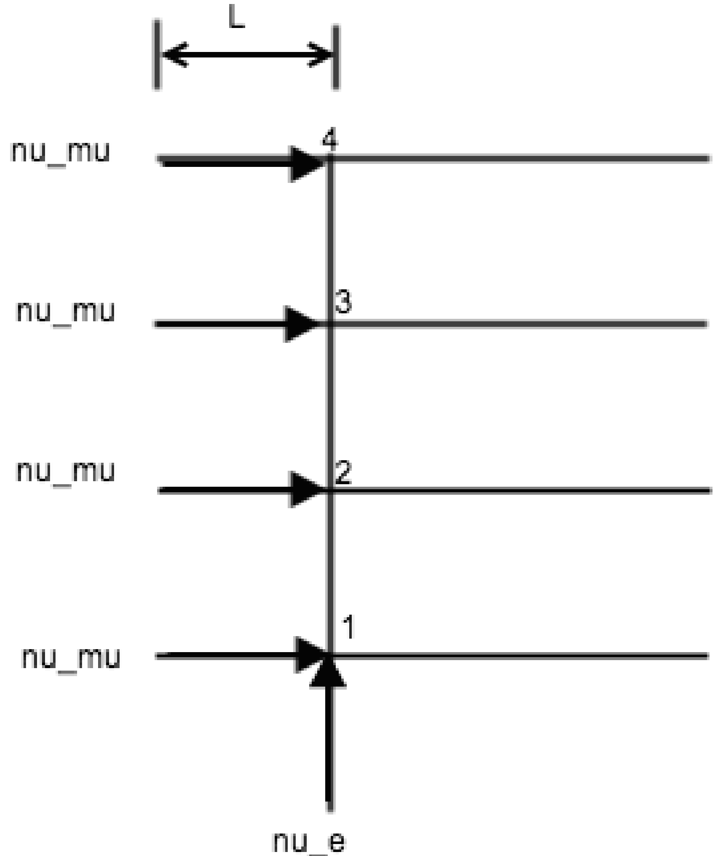


Fig. 9.2: Toy model

At site 1, electron neutrino becomes muon neutrino after 1 oscillation length and moving top, while the muon neutrino coming from the left becomes electron neutrino. If they interact, their momenta will be exchanged, leaving a muon neutrino moving to the right and carrying the momentum of the neutrino moving up.

After the interaction at site 1, a electron neutrino is moving up and transforms to a muon neutrino at site 2. The interaction at site 1 will be repeated all the way along the trajectory. And we have all muon neutrinos coming out right of the sites which should be electron neutrinos if we only have vacuum oscillation.

This is a toy model of collective oscillation.

Spectral Split

A spectral split phenomenon has been observed in calculations.[1]

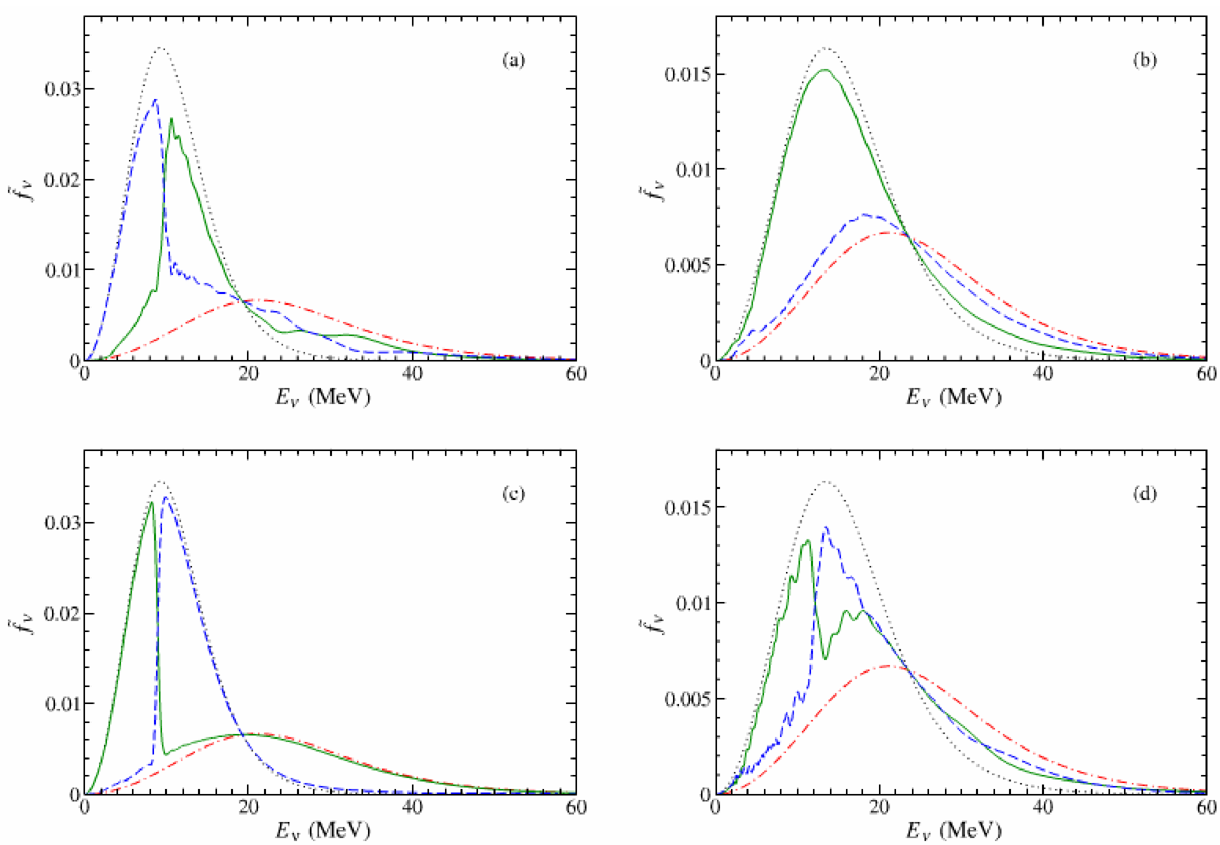


FIG. 7 (color online). Change of energy spectra of neutrinos (left panels) and antineutrinos (right panels) with the normal (upper panels) and inverted (lower panels) neutrino mass hierarchies. The dotted and dot-dashed lines are the spectra of neutrinos (antineutrinos) in the electron and tau flavors, respectively, at $r = R_\nu$, and the solid and dashed lines are the corresponding spectra at $r = 250$ km.

Fig. 9.3: Spectral split due to neutrino self interaction. Total flavour content is not changed however the flavour exchange momentum which is refered to spectral split.

[1] Duan, H., Fuller, G., Carlson, J. & Qian, Y.-Z. Simulation of coherent nonlinear neutrino flavor transformation in the supernova environment: Correlated neutrino trajectories. *Phys. Rev. D* **74**, (2006).

Bipolar Model

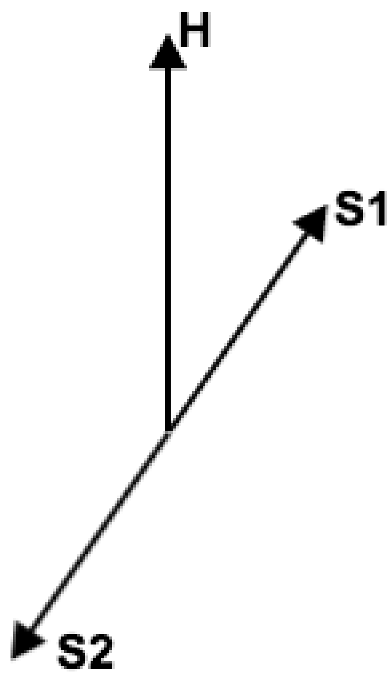


Fig. 9.4: Biplar

The neutrinos are generated in two classes with the same number density thus making up two total flavour isospins. Neutrino-neutrino interaction could make the oscillation unstable if it is too large.[2]

Dense Homogeneous Isotropic Neutrino Gas

The total flavour isospin could precess around effective hamiltonian like the precession of gyroscope with all the indvidual flavour isospin precess around the total flavour isospin.

Refs & Notes

Some papers:

1. Collective neutrino flavor transformation in supernovae

[2] Raffelt, G. & Smirnov, A. Self-induced spectral splits in supernova neutrino fluxes. *Phys. Rev. D* **76**, (2007).

CHAPTER 10

Qualitative Analysis

CHAPTER 11

Instability

Instability of neutrino oscillation means the rapid growth of the oscillations.

Question

Where do we get the perturbations?

Answer

TBD.

Linear Stability Analysis

Bimodal Instability

An example of such intability happens in a system composed of equal amounts of neutrinos and antineutrinos. Flavour transform occurs due to

$$\nu_e + \bar{\nu_e} \leftrightarrow \nu_x + \bar{\nu_x}.$$

Vacuum mixing angle triggers the flavour instability.

Neutrino oscillatioins are synchronized but with a small amplitude inside a SN core (suppressed by matter effects),[1] which basically pin down the flavour transformation. As the flux reaches

[1] Wolfenstein, L. Neutrino oscillations in matter. *Phys. Rev. D* **17**, 23692374 (1978). Or check papers of MSW effect such as Wick Haxton's excellent review.

Multi-angle Instability

Non-isotropic neutrino gas would have velocity (or momentum) related interactions, $1-\vec{v}_p\cdot\vec{v}_q$, which is in fact a $1-\frac{2\sqrt{\pi}}{\sqrt{3}}Y_1^0(\theta,\phi)$ term.

A small anisotropy leads to a runaway flavor equipartition.[2]

MAA

Multi-azimuth angle (MAA) instability, first discovered by Georg Raffelt et al, in the work Axial Symmetry Breaking in Self-Induced Flavor Conversion of Supernova Neutrino Fluxes ,[3] is an intrinsic symmetry breaking. The point is to allow angle modes to evolve independently.

This instability may come from the term that is related to the velocity of neutrinos in the Hamiltonian.

This could happen even for a perfectly symmetric emission.

Neutrino Self Interaction and Instability

Refs & Notes

.

[2] Raffelt, G. & Smirnov, A. Self-induced spectral splits in supernova neutrino fluxes. *Phys. Rev. D* **76**, (2007). In this paper the author adds a small perturbation to a perfectly isotropic neutrino antineutrino gas. The results show multi-angle instability.

[3] Raffelt, G., Sarikas, S. & Seixas, D. *Axial Symmetry Breaking in Self-Induced Flavor Conversionof Supernova Neutrino Fluxes. <http://journals.aps.org/prl/abstract/10.1103/PhysRevLett.111.091101> Phys. Rev. Lett.* **111**, (2013).

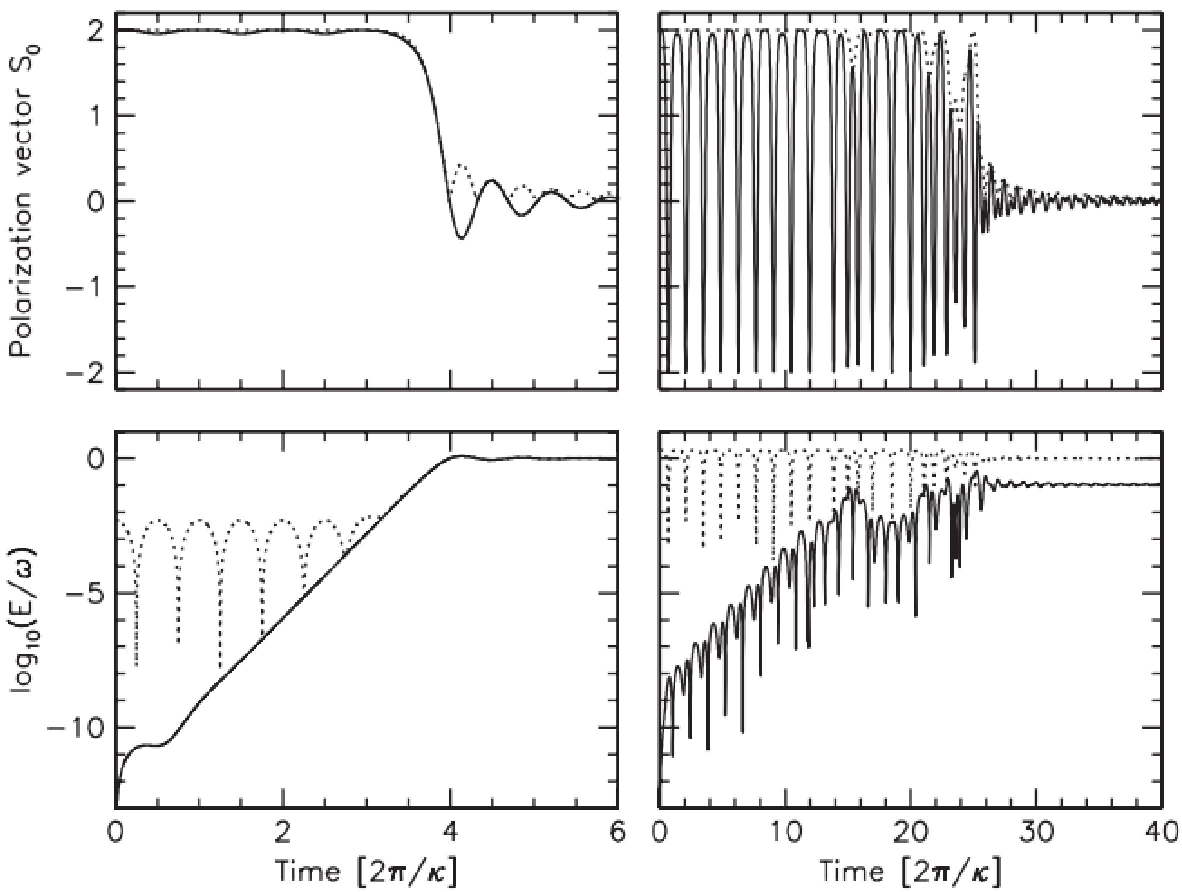


FIG. 10. Evolution of a homogeneous and near-isotropic ensemble of equal ν and $\bar{\nu}$ densities with $\sin 2\Theta = 0.1$ as in Figs. 2 and 3. The initial anisotropy is $\xi = 10^{-4}$. *Top:* polarization vector S_0^z (solid line) and the length $|\mathbf{S}_0|$ (dotted line). *Bottom:* neutrino-neutrino flux energy $-E_1$ (solid line) and potential energy E_ω (dotted line). *Left:* normal hierarchy. *Right:* inverted hierarchy.

Fig. 11.1: A figure from Raffelt & Simirnov (2007) shows the instability from anisotropic small perturbations. Potential energy grows expotentially, where $-E_1 = \mu/4\vec{D}_1^{\,2}$.

CHAPTER 12

Pictures

There are several pictures to visualize the oscillations of neutrinos.

Magnetic Spin

Fig. 12.1: Image source: Larmor Precession .

Recall that torque of a magnetic spin in a magnetic field is calculated as

$$\vec{\tau} = \vec{\mu} \times \vec{B},$$

while torque is by definition $\vec{\tau} = \frac{d}{dt}\vec{L}$. So we have, for such a system, the equation of motion is

$$\frac{d}{dt}\vec{L} = \vec{\mu} \times \vec{B}.$$

In the case of electron quantum magnetic spin, $\vec{\mu}$ is proportional to the angular momentum $\vec{L}$, i.e., $\vec{\mu} = \frac{-e}{2m_e}\vec{L} \propto \vec{L}$. So the equation of motion becomes

$$\frac{d}{dt}\vec{L} \propto \vec{L} \times \vec{B}.$$

Equation of Motion for Neutrino Flavor Polarization Vector

That EoM is

$$\frac{d}{dt}\vec{P}_\omega = (\omega\vec{B} + \lambda\vec{L} + \mu\vec{D}) \times \vec{P}_\omega,$$

where the quantities can be found in Duan, H., Fuller, G. & Qian, Y.-Z. Collective Neutrino Oscillations. *Annu. Rev. Nucl. Part. Sci.* **60**, 569–594 (2010).

Now it is clear that the two system has very similar EoM.

Neutrino Flavour Isospin

Neutrino flavour isospin[3]

$$\mathbf{s} = \psi^{f\dagger}\frac{\sigma}{2}\psi^f,$$

where

$$\psi^f_\nu = \begin{pmatrix} a_{\nu_e} \\ a_{\nu_x} \end{pmatrix}$$
$$\psi^f_{\bar\nu} = \begin{pmatrix} -a_{\bar\nu_x} \\ a_{\bar\nu_e} \end{pmatrix}$$

The equation of motion for isospin is

$$\frac{d}{dt}\mathbf{s} = \mathbf{s}\times\mathbf{H}^{\mathbf{eff}}.$$

Previously we have already seen the equations for a spinning top,

$$\frac{d}{dt}\vec{S} = \frac{\partial}{\partial t}\vec{S} - \vec{S}\times\vec{\Omega},$$

where $\vec{\Omega} = \vec{n}\dot{\phi}$. Consider conservation of momentum, we have

$$\frac{\partial}{\partial t}\vec{S} = \vec{S}\times\vec{\Omega},$$

which is similar to the neutrino isospin equation of motion. $\vec{\Omega}$ corresponds to $\mathbf{H}^{\mathbf{eff}}$.

Coupled Pendulum

The equation of motion is

$$\begin{pmatrix} -\frac{d^2}{dt^2} - \left(\frac{g}{l}+\frac{k}{m}\right) & \frac{k}{m} \\ \frac{k}{m} & -\frac{d^2}{dt^2} - \left(\frac{g}{l}+\frac{k}{m}\right) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using Fourier transform, we will get the solutions,

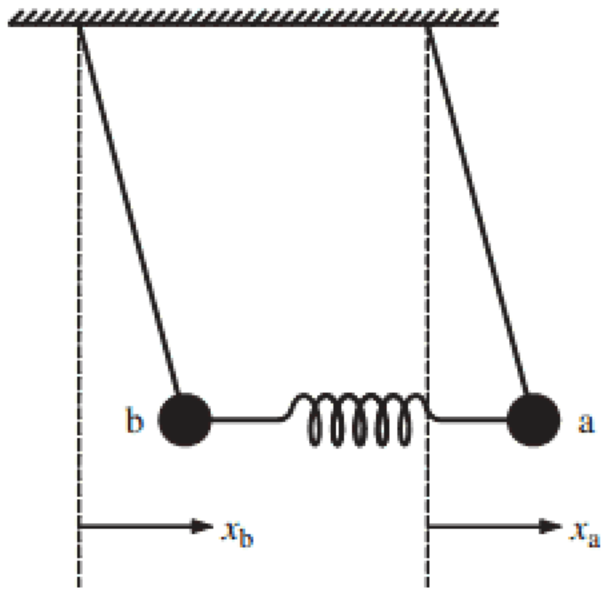
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_1\cos(\omega_1 t+\phi_1) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} A_2\cos(\omega_2 t+\phi_2)$$

Recall that the state of neutrino after time t is

$$|\psi(t)\rangle = A_1|\nu_1\rangle e^{-iE_1t} + A_2|\nu_2\rangle e^{-iE_2t},$$

where A_1 and A_2 are determined by initial condition. The real part of this, is exactly the same as the solution to coupled pendulum, where the physics is the transfer from one eigenstate to another.

[3] Collective neutrino flavor transformation in supernovae



Gyroscope or Spinning Top Picture

A Classical Top

The key concept of a classical gyroscope is the balance between gravity and angular momentum conservation, i.e., angular conservation in specific directions.

Angular momentum for a 3D rigid body with a axial symmetry in a $\dot{\vec{I}} = 0$ frame is

$$\vec{S} \to \begin{pmatrix} I_0\omega_x \\ I_0\omega_y \\ I\omega_z \end{pmatrix}$$

The gyroscope should obey Euler's equations with extra Coriolis terms since we have decided to work in a rotation frame ($\dot{\vec{I}} = 0$),[1]

$$\begin{aligned} M_x &= I_0(\ddot{\theta} - \dot{\phi}^2 \sin\theta\cos\theta) + I\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi}) \\ M_y &= I_0(\ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta) - I\dot{\theta}(\dot{\phi}\cos\theta + \dot{\psi}) \\ M_z &= I(\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta) \end{aligned}$$

with the torque for a top being

$$\begin{aligned} M_x &= mgz_G\sin\theta \\ M_y &= 0 \\ M_z &= 0. \end{aligned}$$

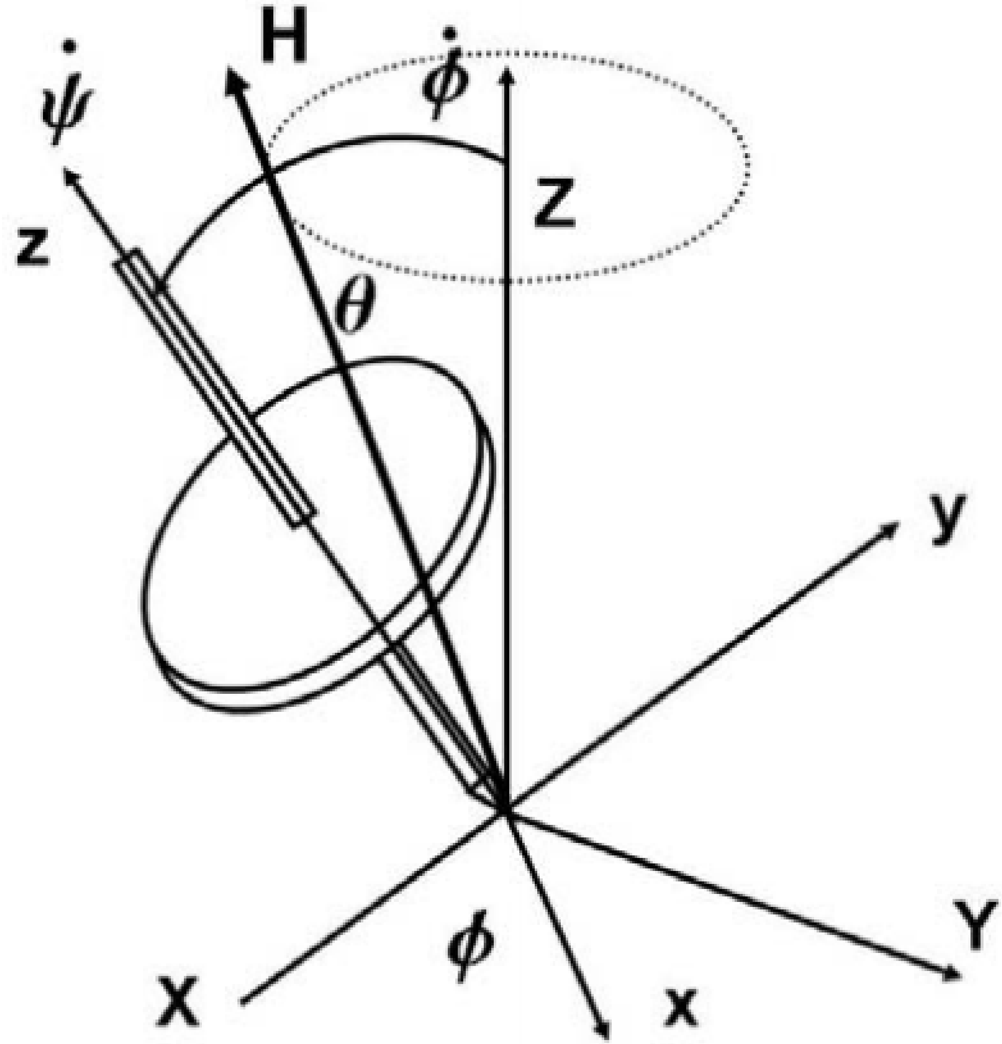
Note: $\dot{\psi}$ is the spin of the top itself. More generally, the Euler equation is

$$\mathbf{I}\cdot\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times(\mathbf{I}\cdot\boldsymbol{\omega}) = \mathbf{M}.$$

[1] Read Carl's lecture notes of *Classical Mechanics* for this derivation.

Steady Precession

A steady precession maintains the angle θ.



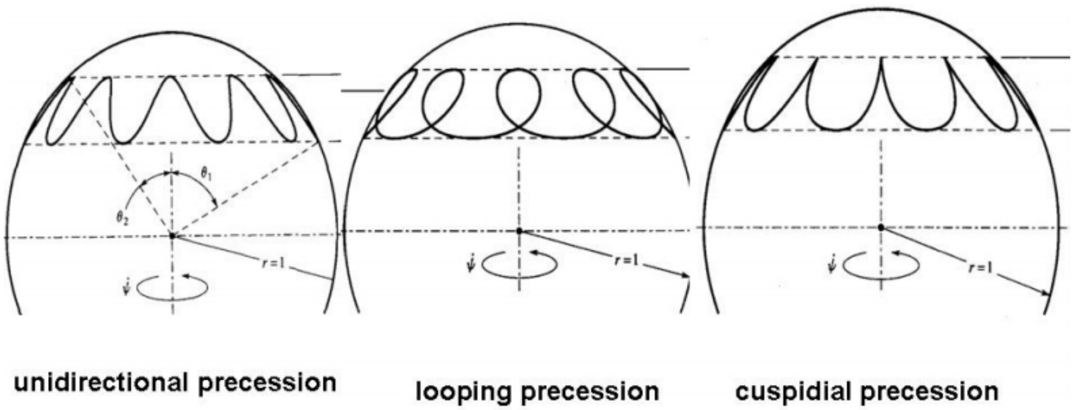
Now we have $\dot{\theta} = 0$ so the Euler's equations reduces to,

$$\dot{\phi}(I(\dot{\phi}\cos\theta + \dot{\psi}) - I_0\dot{\phi}\cos\theta) = M_x \equiv mgz_G$$

For a steady state usually we can use this approximation $\dot{\psi} \gg \dot{\phi}$.

$$\dot{\phi} = \frac{mgz_G}{I\dot{\psi}}.$$

Now define $\Omega = \dot{\phi}$ and $\omega = \dot{\psi}$. Our approximation becomes $\omega \gg \Omega$.



Unsteady Precession

Polarization Vector

Polarization (for a two state system) is the difference of the probabilities of finding the system in the two difference normal states (spin up and spin down for example).

Density Matrix

For a two-state system, an example of density matrix is

$$\hat{\rho} = W_1 \left|\psi_a\right\rangle \left\langle\psi_a\right| + W_2 \left|\psi_b\right\rangle \left\langle\psi_b\right| .$$

When $\{\left|\psi_a\right\rangle, \left|\psi_b\right\rangle\}$ basis is chosen, density matrix can be written as a matrix,

$$\rho = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix},$$

in which the two constants are the probability to find the system in each states respectively and they are called the population.

Rewrite the density matrix with Pauli matrices and identity,

$$\rho = \frac{1}{2}(\mathbf{I} + \vec{\sigma}\vec{P}).$$

Note: The reason we have a $\frac{1}{2}$ is that by definition polarization vector is

$$\begin{aligned} \rho &= a_0\mathbf{I} + \sigma_{\mathbf{x}} a_x + \sigma_{\mathbf{y}} a_y + \sigma_{\mathbf{z}} a_z \\ &= a_0\mathbf{I} + \vec{a}\vec{\sigma}. \end{aligned}$$

However, trace of density matrix should be 1, which means $\mathrm{Tr}\rho = a_0 2 = 1$ and we can find $a_0 = \frac{1}{2}$ noting that $\mathrm{Tr}\sigma_i = 0$.

The important fact is that the values of polarization depends on the choice of basis.

More physical meanings can be obtained by chosing a good basis so that the density matrix is diagonalised by expressing it with components of polarization.[4]

[4] Read quantum statistics book if more is needed.

Polarization, as the name indicates, should equal to

$$P = W_1 - W_2$$

when it is aligned with z direction of Pauli matrices. Polarization vector is not a vector in real space but a vector of an imagined space.

Take Out The Components

How to project out the components of polarization vector? By multiplying on both sides the Pauli matrices.

Note that for Pauli matrices

$$\sigma_i \sigma_j = \epsilon_{ijk} \sigma_k + \delta_{ij} \mathbf{I}.$$

Multiplying by σ_j on both sides of $\rho = \frac{1}{2}(\mathbf{I} + \vec{P} \cdot \vec{\sigma})$, we get

$$\rho \sigma_j = \frac{1}{2}(\mathbf{I} + \vec{P} \cdot \vec{\sigma}) \sigma_k.$$

Apply the sigma algebra we discussed there, the result of this is

$$\rho \sigma_j = \frac{1}{2}(\sigma_k + P_i \sigma_i \sigma_j)$$
$$2\rho \sigma_j - \sigma_k = P_i(\epsilon_{ijk} \sigma_k + \delta_{ij} \mathbf{I})$$

We know that the trace of any Pauli matrix is zero. Take the trace of the equation,

$$\mathrm{Tr} \sigma_j = P_j.$$

All done.

Neutrino-neutrino Interaction and BCS Theory

BCS

BCS Hamiltonian is

$$\hat{H}_{BCS} = \sum_k 2\epsilon_k \hat{t}_k^0 - |G| \hat{T}^\dagger \hat{T}$$

Neutrino self interaction Hamiltonian is

$$\hat{H} = \sum_p \frac{\delta^2 m}{2p} \hat{B} \cdot \vec{J}_\rho | \frac{\sqrt{2} G_F}{V} \vec{J} \cdot \vec{J}.$$

CHAPTER 13

Models

Homogeneous and Isotropic Neutrino Gas

$\vec{P}_\omega$ can be time-independent.

Regarding the discussion in Pictures, we know immediately that $\vec{H}$ is aligned or anti-aligned with $\vec{P}_\omega$ since

$$\vec{H} \times \vec{P}_\omega = 0,$$

as $\vec{P}_\omega$ is time-independent.

Anyway the equation we need to solve is then

$$\vec{a}$$

What to expect?

Without solving the equation, we know that

Solving Eqns

CHAPTER 14

Neutrino Oscillation And Master Equation

Question

Why do we think about master equation?

Answer

The terms we care the most are the populations of the states. One of the treatment of quantum master equation is to write down the closed equations for population terms only. A very beautiful example is the projection method invented by Zwawzig and Nakajiwa.

WHY

Why do you need a master equation approach? IDK.

Quantum Master Equation

Projection Technique

First of all define a diagonalizing operator $\hat{D}$ which just keeps the diagonal elements and simply drops the off diagonal elements. We see that $1 - \hat{D}$ will element all diagonal elements.

We can define the diagonalized density matrix as $\hat{\rho}_d = \hat{D}\hat{\rho}$ and off-diagonalized density matrix as $\hat{\rho}_{od} = (1 - \hat{D})\hat{\rho}$. As an application,

$$\hat{\rho} = \hat{\rho}_d + \hat{\rho}_{od}.$$

Starting from the von Neumann equation,

$$i\hbar\partial_t\hat{\rho} = \left[\hat{H}, \hat{\rho}\right].$$

By using the Liouville operator,

$$\partial_t\hat{\rho} = -i\hat{L}\hat{\rho}.$$

Apply $\hat{D}$ and $1-\hat{D}$ to the von Neumann equation,

$$\partial_t\hat{\rho}_d = -i\hat{D}\hat{L}\hat{\rho}$$
$$\partial_t\hat{\rho}_{od} = -i(1-\hat{D})\hat{L}\hat{\rho}.$$

Use the relation that $\hat{\rho} = \hat{\rho}_d + \hat{\rho}_{od}$, we have

$$\partial_t\hat{\rho}_d = -i\hat{D}\hat{L}\hat{\rho}_d - i\hat{D}\hat{L}\hat{\rho}_{od}$$
$$\partial_t\hat{\rho}_{od} = -i(1-\hat{D})\hat{L}\hat{\rho}_d - i(1-\hat{D})\hat{L}\hat{\rho}_{od}.$$

Solve the second equation using Green function technique,

$$\hat{\rho}_{od} = e^{-i(1-\hat{D})\hat{L}t} + \int_0^t dt' e^{-i(1-\hat{D})\hat{L}(t-t')}(-i(1-\hat{D})\hat{L}\hat{\rho}_d(t')).$$

Hint: Recall that the solution for

$$\dot{y} + \alpha y = f$$

is

$$y = e^{-\alpha t}y(0) + \int_0^t dt' e^{-\alpha(t-t')}f(t').$$

Insert this solution to the equation of $\hat{\rho}_d$,

$$\partial_t\hat{\rho}_d = -i\hat{D}\hat{L}\hat{\rho}_d - \hat{D}\hat{L}\int_0^t dt' e^{-i(1-\hat{D})\hat{L}(t-t')}(1-\hat{D})\hat{L}\hat{\rho}_d(t') - i\hat{D}\hat{L}e^{-i(1-\hat{D})\hat{L}t}\rho_{od}(0).$$

What happened to the blue term? It disapears when we apply the initial random phase condition.

When it happens we get our closed master equation for $\hat{\rho}_d$, which is an equation for the probability.

In our case of neutrinos, random phase condition is not really needed since we usually deal with the situation that electron neutrinos are appearant only.

In details, we have such a density matrix,

$$\rho = \begin{pmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ba} & \rho_{bb} \end{pmatrix}.$$

The quantum master equation we would like to use is

$$\partial_t\hat{\rho}_d = -i\hat{D}\hat{L}\hat{\rho}_d - \hat{D}\hat{L}\int_0^t dt' e^{-i(1-\hat{D})\hat{L}(t-t')}(1-\hat{D})\hat{L}\hat{\rho}_d(t').$$

Vacuum Oscillation Master Equation

Using this projection method, one can find out the master equation for vacuum oscillations.

Pauli Matrices

We will use Pauli matrices in the following part. Here is a review of them.

1. Pauli Matrices,

$$\begin{aligned}\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\end{aligned}$$

2. Commutation Relations,

$$\begin{aligned}[\sigma_1, \sigma_2] &= 2i\sigma_3 \\ [\sigma_2, \sigma_3] &= 2i\sigma_1 \\ [\sigma_3, \sigma_1] &= 2i\sigma_2.\end{aligned}$$

The general form is

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k.$$

All the Pauli matrices plus identity form a complate basis for 2 by 2 matrices. Vacuum oscillation Hamiltonian is

$$\begin{aligned}\mathbf{H} \to & \frac{\delta^2 m}{4E} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \\ \equiv & \begin{pmatrix} -c & s \\ s & c \end{pmatrix} \\ = & -c \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + s \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ = & -c\boldsymbol{\sigma_3} + s\boldsymbol{\sigma_1},\end{aligned}$$

where $c \equiv \frac{\delta^2 m}{4E}\cos 2\theta$ and similarly for s.

Liouville Operator

Liouville operator in quantum mechanics is

$$\hat{L} = [H, *],$$

where the asterisk is the slot for an operator.

In the case of vacuum oscillation, we can calculate the following results,

$$\begin{aligned}\hat{L}\sigma_1 &= [H, \sigma_1] = -2ic\sigma_2 \\ \hat{L}\sigma_2 &= [H, \sigma_2] = 2ic\sigma_1 + 2is\sigma_3.\end{aligned}$$

Notice that σ_3 has diagonal terms only. It will dispear when we apply $1-\mathscr{D}$ which removes the diagonal elements, i.e.,

$$(1-\mathscr{D})\hat{L}\sigma_1 = -2ic\sigma_2$$
$$(1-\mathscr{D})\hat{L}\sigma_2 = 2ic\sigma_1.$$

Diagonalized density matrix $\rho_d = \mathrm{diag}(\rho_1, \rho_2)$ is

$$\begin{aligned}\rho_{\mathrm{d}} &= \begin{pmatrix}\rho_1 & 0\\ 0 & \rho_2\end{pmatrix}\\ &= \frac{1}{2}\left(\begin{pmatrix}\rho_1-\rho_2 & 0\\ 0 & \rho_2-\rho_1\end{pmatrix} + \begin{pmatrix}\rho_1+\rho_2 & 0\\ 0 & \rho_1+\rho_2\end{pmatrix}\right)\\ &= \frac{1}{2}\left((\rho_1-\rho_2)\sigma_3 + (\rho_1+\rho_2)\mathbf{I}\right)\end{aligned}$$

Note: Actually $\rho_1+\rho_2 = 1$ for such a system. We'll see the proof of this later.

Apply $(1-\mathscr{D})\hat{L}$ we get

$$\begin{aligned}(1-\mathscr{D})\hat{L}\rho_d &= is(\rho_2-\rho_1)\sigma_2,\\ \mathscr{D}\hat{L}\rho_d &= -\frac{1}{2}c(\rho_1+\rho_2)\sigma_3.\end{aligned}$$

Exponential Operator

Exponential operator is understood when series expansion is done,

$$e^{\hat{A}} = \hat{I} + \hat{A} + \frac{1}{2!}\hat{A}^2 + \frac{1}{3!}\hat{A}^3 + \cdots$$

Recall that the master equation is

$$\begin{aligned}\partial_t\rho_d(t) &= -i\mathscr{D}\hat{L}\rho_d - \mathscr{D}\hat{L}\int_0^t dt' e^{-i(1-\mathscr{D})\hat{L}(t-t')}(1-\mathscr{D})\hat{L}\hat{\rho}_d(t')\\ &= \frac{1}{2}ic(\rho_1+\rho_2)\sigma_3 - \mathscr{D}\hat{L}\int_0^t dt'\left(is(\rho_2-\rho_1)e^{-i(1-\mathscr{D})\hat{L}(t-t')}\sigma_2\right)\end{aligned}$$

So we need to calculate

$$\begin{aligned}e^{-i(1-\mathscr{D})\hat{L}(t-t')}\sigma_2 &= \left[1 - i(1-\mathscr{D})\hat{L}(t-t') + \frac{1}{2}(-i(1-\mathscr{D})\hat{L}(t-t'))^2 + \frac{1}{3!}(-i(1-\mathscr{D})\hat{L}(t-t'))^3 + \cdots\right]\sigma_2\\ &\equiv T_0 + T_1 + \frac{1}{2}T_2 + \frac{1}{3!}T_3 + \cdots.\end{aligned}$$

We will calculate it term by term and find the pattern.

$$T_0 = \sigma_2$$

$$\begin{aligned}T_1 &= -i(1-\mathscr{D})\hat{L}(t-t')\sigma_2\\ &= 2c\sigma_1(t-t')\end{aligned}$$

$$\begin{aligned}T_2 &= -i(1-\mathscr{D})\hat{L}(t-t')(2c\sigma_1(t-t'))\\ &= -i(t-t')^2 2c(-2ic\sigma_2)\\ &= -2^2c^2(t-t')^2\sigma_2\end{aligned}$$

$$\begin{aligned}T_3 &= -i(1-\mathscr{D})\hat{L}(t-t')(-4c^2(t-t')^2\sigma_2)\\ &= -2^3c^3(t-t')^3\sigma_1\end{aligned}$$

$$\begin{aligned}T_4 &= -i(1-\mathscr{D})\hat{L}(t-t')(-2^3c^3(t-t')^3\sigma_1)\\ &= -i(t-t')(-2^3c^3(t-t')^3)(-2ic\sigma_2)\\ &= 2^4c^4(t-t')^4\sigma_2\end{aligned}$$

$$\begin{aligned}T_5 &= -i(1-\mathscr{D})\hat{L}(t-t')2^4c^4(t-t')^4\sigma_2\\ &= -i(t-t')2^4c^4(t-t')^4 2ic\sigma_1\\ &= 2^5c^5(t-t')^5\sigma_1\end{aligned}$$

Carry on this calculation we can infer that

$$e^{-i(1-\mathscr{D})\hat{L}(t-t')}\sigma_2 = \sigma_2 + 2c\sigma_1(t-t') + \frac{1}{2}(-2^2c^2(t-t')^2\sigma_2) + \frac{1}{3!}(-2^3c^3(t-t')^3\sigma_1) + \frac{1}{4!}2^4c^4(t-t')^4\sigma_2 + \frac{1}{5!}2^5c^5(t-t')^5\sigma_1 +$$

Taylor Series

Taylor series of $\sin x$ and $\cos x$ around $x = 0$ are

$$\begin{aligned}\sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\\ \cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots .\end{aligned}$$

Now we see that

$$e^{-i(1-\mathscr{D})\hat{L}(t-t')}\sigma_2 = \sigma_1\sin(M) + \sigma_2\cos(M),$$

where $M \equiv 2c(t-t')$.

The master equation we need is

$$\begin{aligned}\partial_t\rho_d(t) &= \frac{1}{2}ic(\rho_1+\rho_2)\sigma_3 - \mathscr{D}\hat{L}\int_0^t dt' is(\rho_2-\rho_1)\left(\sigma_1\sin(2c(t-t')) + \sigma_2\cos(2c(t-t'))\right)\\ &= \frac{1}{2}ic(\rho_1+\rho_2)\sigma_3 - \mathscr{D}\hat{L}is\int_0^t dt'(\rho_2-\rho_1)\left(\sigma_2\cos(2c(t-t'))\right)\\ &= \frac{1}{2}ic(\rho_1+\rho_2)\sigma_3 - isG(t)\mathscr{D}\hat{L}\sigma_2\\ &= \frac{1}{2}ic(\rho_1+\rho_2)\sigma_3 + 2s^2G(t)\sigma_3\\ &= \frac{1}{2}ic(\rho_1+\rho_2)\sigma_3 + 2s^2\int_0^t dt'(\rho_2-\rho_1)\sigma_3\cos(2c(t-t'))\\ &= \frac{1}{2}ic(\rho_1+\rho_2)\sigma_3 + 2s^2\int_0^t dt'\left(-2\rho_d(t') + (\rho_1+\rho_2)\mathbf{I}\right)\cos(2c(t-t'))\\ &= \frac{1}{2}ic\sigma_3 + 2s^2\int_0^t dt'\left(-2\rho_d(t') + \mathbf{I}\right)\cos(2c(t-t'))\end{aligned}$$

In the calculation, $G = \int_0^t dt'(\rho_2-\rho_1)\cos(2c(t-t'))$.

What to Do?

I don't see anything good about this method. What to do next? I can predict that it's also won't cost a lot to solve the MSW effect. But what's the point? These problems are not very hard to solve even using wave function method.

I am just leaving this result here and move on to other topics.

Neutrino Oscillation in Matter - A Possible Master Equation Approach

Self Interaction Between Neutrinos

The neutrino-neutrino interaction Hamiltonian involves the density matrix, which makes it very hard to find a closed equation.

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CHAPTER 15

Effect of Gravitation

The effect of gravitation on neutrino oscillation could be great around a neutron star.

The spacetime are distorted around the neutron star.

- Time gradient/delay; Shapiro delay;
- Space geometry/trajectory; self-interaction $\vec{v} \cdot \vec{v}'$;
- Redshift;
- Tidal effect;
- Coupling of space and time due to cross terms (might need quantum field theory in curved spacetime);
- Lense-Thirring effect.

Evaluation

The equation of motion in a linear approximation (with respect to κ) is

$$\frac{d}{d\tau} u_\mu + \left(\kappa h_{\mu\alpha,\beta} u^\alpha u^\beta - \frac{\kappa}{2} h_{\alpha\beta,\mu} u^\alpha u^\beta \right) = 0,$$

where $\kappa = 8\pi G$, $h_{\alpha\beta}$ is the metric tensor of the gravitational field, that is,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \kappa h_\alpha.$$

Another useful form is to multiply on both side $md\tau$ and substitute mass terms with 4-momentum $p_\mu = mu_\mu$.

$$dp_\mu + \left(\kappa h_{\mu\alpha,\beta} p^\alpha - \frac{\kappa}{2} h_{\alpha\beta,\mu} p^\alpha \right) dx^\beta = 0$$

Deflection in The Trajectory

For the trajectory of photons, we only need to find out the geodesic. Neutrinos are massive particles which are different from photons. However, the neutrinos we are considering have energy as high as several MeVs or even more while their mass are less than $1\mathrm{eV}$. In this case, they are relativistic so their trajectory are close to photons'.

The deflection of photons near a star is,

$$\delta \approx -\frac{4GM_\odot}{bc^2}.$$

Derivation

Suppose we have a photon coming along z axis from infinite, the deflected angle at infinite is, by first order approximation $\tan\delta \approx \delta$, the change of momentum in x direction over the momentum in z direction,

$$\delta \approx \frac{\Delta p_x}{p_z}.$$

Momentum in this coordinate system is

$$p^\alpha = (cp^3, 0, 0, p^3).$$

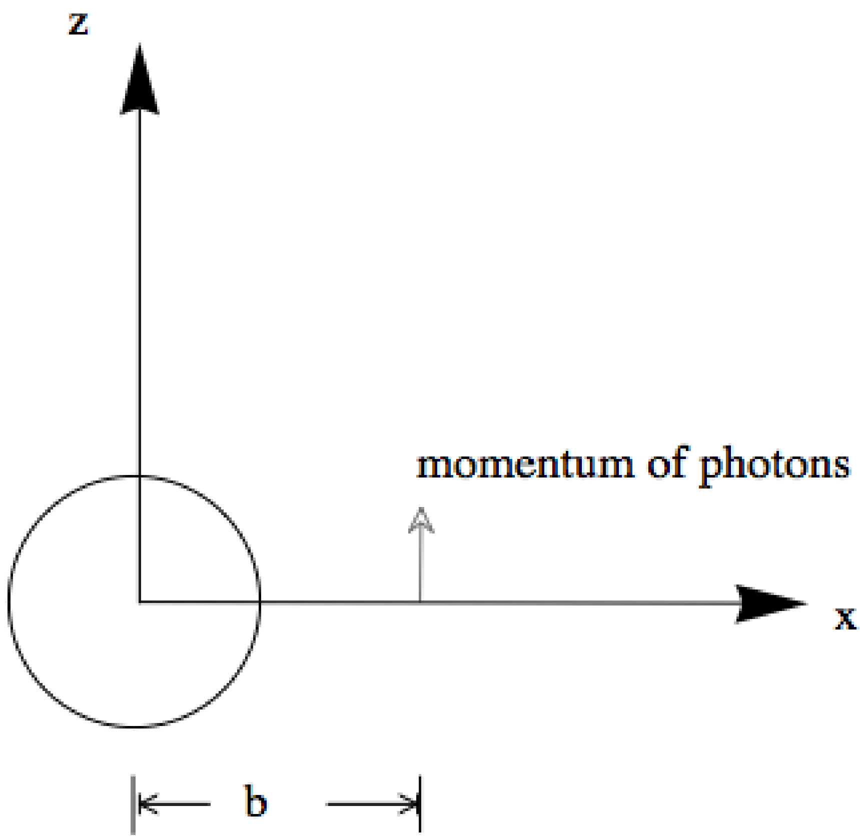
Displacement is

$$dx^\beta = (dz/c, 0, 0, dz).$$

Then the change in momentum can be calculated using the equation of motion,

$$\Delta p_1 = -\kappa p^\alpha \int_{-\infty}^{\infty} \left(h_{1\alpha,\beta} - \frac{1}{2} h_{\alpha\beta,1} \right) dx^\beta.$$

This is the deflection angle of a photon coming from infinite. However, the angle deflected for a photon emitted at tangent is different.



A detailed calculation shows,[1]

With an impact parameter of $b = 10$km, the angle will eventually become larger than 0.4, which is very significant.

Refs & Notes

Here is a list of papers on the gravitational effects of neutrino oscillations,

1. Gravitational Effects on the Neutrino Oscillation
2. Neutrino oscillations in curved spacetime: an heuristic treatment
3. Neutrino Oscillations in Gravitational Field
4. Neutrino oscillations in Kerr-Newman space-time
5. Neutrino oscillations in strong gravitational fields by Dardo Píriz, Mou Roy, and José Wudka
6. Can Gravity Distinguish Between Dirac and Majorana Neutrinos? (on PRL)

And related topics

1. A comparison between matter wave and light wave interferometers for the detection of gravitational waves
2. Matter waves in a gravitational field: An index of refraction for massive particles in general relativity

[1] The MMA file is here .

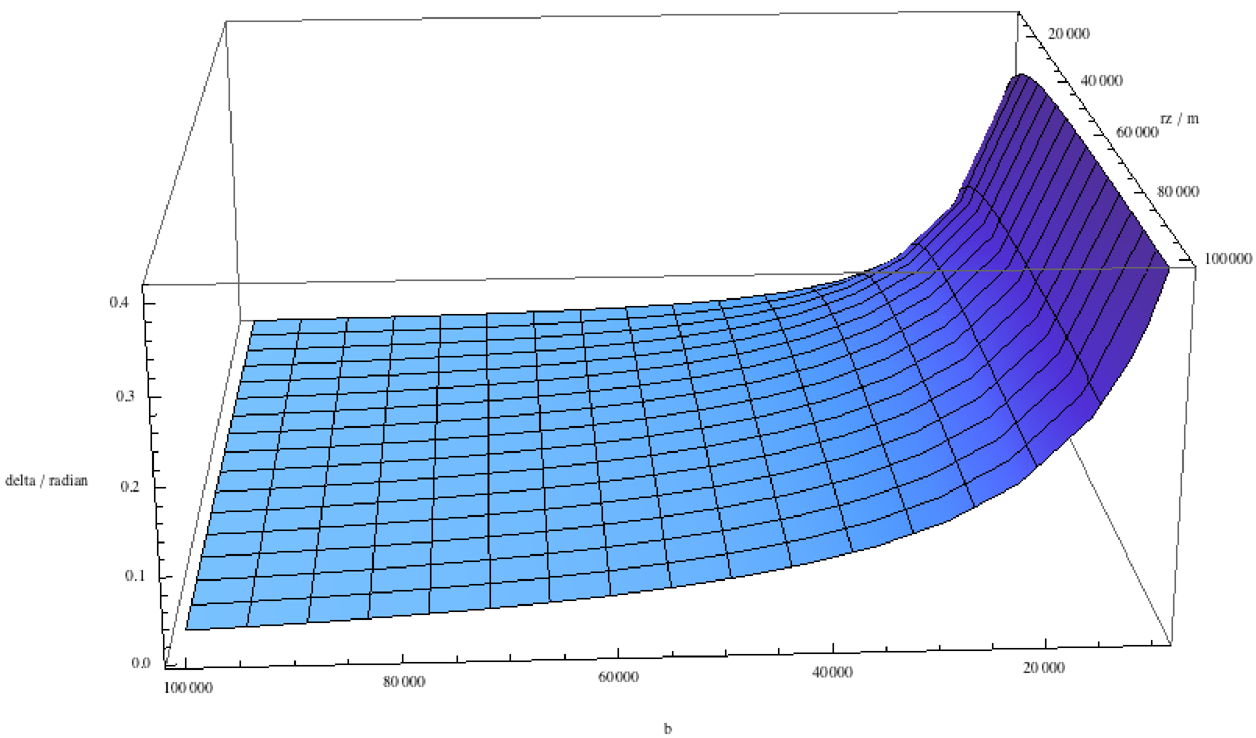


Fig. 15.1: The deflection angle of a photon starting from a tangent position at $z = rz$ with tangent momentum and impact parameter b.

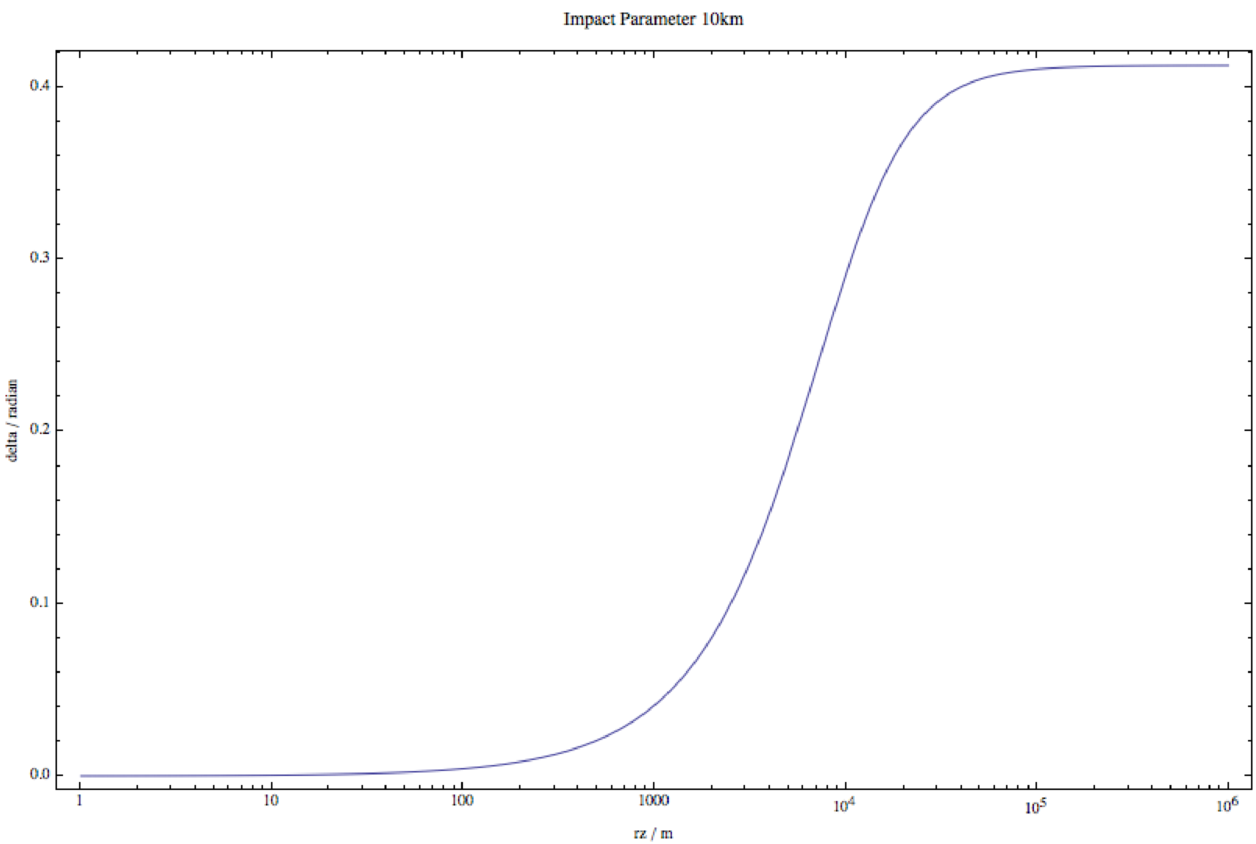


Fig. 15.2: As we can see the angle becomes very big at about 10km.

CHAPTER 16

References

Wick C. Haxton & Barry R. Holstein wrote two good reviews of neutrino physics,

- Neutrino Physics
- Neutrino Physics: an Update

A RMP review by S. M. Bilenky and S. T. Petcov

- Massive neutrinos and neutrino oscillations

PDG has a complex review of neutrino mass problems

1. NEUTRINO MASS, MIXING, AND OSCILLATIONS

About coherence:

1. Coherence of Nuetrino Oscillations in The Wave Packet Approach by Giunti & Kim. They gave a simple but elegent way to calculate the decohrence.

Review of Collective Oscillations

- Duan, H., Fuller, G. M., & Qian, Y.-Z. (2010). Collective Neutrino Oscillations. doi:10.1146/annurev.nucl.012809.104524

Raffelt has a paper about axial symmetry breaking here.

- Raffelt, G., Sarikas, S., & Seixas, D. D. S. (2013). Axial Symmetry Breaking in Self-Induced Flavor Conversion of Supernova Neutrino Fluxes, 091101(August), 1–5. doi:10.1103/PhysRevLett.111.091101

Boris Kayser wrote a bad typesetted paper explaining the fundamental question of neutrinos.

- Kayser, B. (2009). Are Neutrinos Their Own Antiparticles?, 012013, 8. doi:10.1088/1742-6596/173/1/012013

CHAPTER 17

From Neutrinos to Cosmos

Neutrinos, or rather weak interactions, play a very important role in cosmology.

CHAPTER 18

MISC

Neutrino & Transport

CHAPTER 19

Questions

Question

Is neutrino its own antiparticle? Or is neutrino Majorana or dirac?

Question

What's the mass hierarchy?

Question

What are the mixing angles?

Question

How many different flavours of neutrinos?

CHAPTER 20

Definition

$$
\begin{aligned}
\Delta &= \sqrt{2} G_F n(x) \\
\omega &= \frac{\Delta m^2}{2E}
\end{aligned}
$$

CHAPTER 21

Support

CHAPTER 22

DOI

CHAPTER 23

Footnote

Bibliography

[Parke1986] Parke, S. J. (1986). Nonadiabatic Level Crossing in Resonant Neutrino Oscillations. Physical Review Letters, 57(10), 1275–1278. doi:10.1103/PhysRevLett.57.1275

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