hawkeslib Documentation

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Contents:

hawkeslib, Python library for fast parameter estimation in vanilla Hawkes process models.

CHAPTER 1

Introduction

1.1 About hawkeslib

hawkeslib started with the ambition of presenting easy-to-use, well maintained implementations of plain-vanilla Hawkes (self-exciting) processes[1][2], a form of evolutionary temporal point processes that is increasingly put to use in a variety of domains.

Some features of the library are

- Fast. most algorithms are implemented in Cython, cutting away most of the Python overhead for likelihood computations and parameter estimation algorithms that require *scanning* the data.
- Easy-to-use. Models implement a familiar, common interface.
- Good for beginners. The library implements a variety of *plain vanilla* self-exciting processes such as univariate and multivariate Hawkes processes with exponential delay densities, Poisson processes, and related Bayesian inference machinery.

In the future, we hope to add several extended models.

1.2 Installation

Cython (>=0.28) and numpy (>=1.14) must be installed prior to the installation.

[1] Hawkes, Alan G. "Point spectra of some mutually exciting point processes." Journal of the Royal Statistical Society. Series B (Methodological) (1971): 438-443.

[2] Bacry, Emmanuel, Iacopo Mastromatteo, and Jean-François Muzy. "Hawkes processes in finance." Market Microstructure and Liquidity 1.01 (2015): 1550005.

```
$ pip install -U Cython numpy
$ pip install hawkeslib
```

Currently, the library only supports python 2.7.

1.3 Getting Started

The `examples/` folder contains Jupyter notebooks that demonstrate basic use cases, in addition to the tutorial and example provided in these docs.

References

CHAPTER 2

Tutorial: Hawkes Processes

Here we present an informal introduction to the ideas and theory behind self-exciting (Hawkes) processes. We assume the reader has some familiarity with statistics and machine learning. However, rather than walking the reader through heavy math, we try to highlight the intuition of Hawkes and its potential application areas.

2.1 Some Background

Many natural phenomena of interest, to machine learning as well as other disciplines, include **time** as a central dimension of analysis. A key task, then, is to capture and understand statistical relationships along the timeline. "Workhorse" models addressing *temporal* data are collected under *time-series analysis*. Such models often divide time into equal-sized buckets, associate quantities with each such bucket on which the models operate. This is the *discrete-time* formalism that appears in many models familiar in machine learning such as Kalman filters and hidden Markov models, as well as common models in econometrics and forecasting such as ARIMA or exponential smoothing.

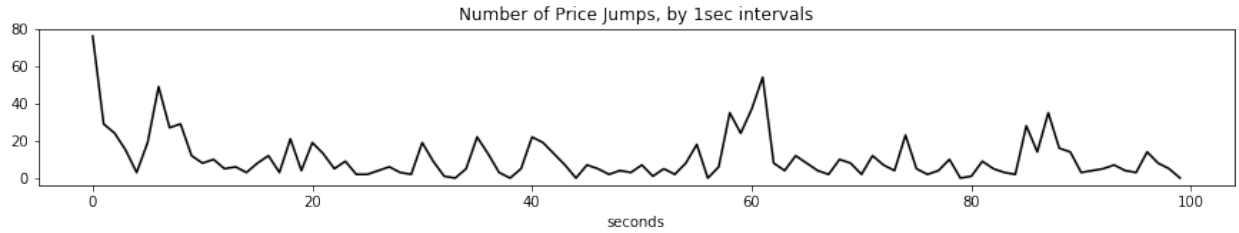
Say we are provided the timestamps of all "price jumps" in a financial market. Exploring this data set from a temporal standpoint, we often take the path of "aggregating" statistics on a uniform time grid, and running models on these aggregates. Concretely, take "millisecond" timestamps of financial market price events:

```
%matplotlib inline
import numpy as np
import pandas as pd
from matplotlib import pyplot as plt

# here are the "timestamps"
df = pd.read_csv("example_data_top4.csv", header=None)
ar = np.array(df.loc[:, 1])

print ar
```

```
[     56      56      59 ... 8199797 8199798 8199984]
```

Below, we aggregate them to 1 second intervals –collecting the number of "events" at each interval– to arrive at a familiar "time-series" plot.

```
# here is the "aggregated" time series plot
bc = np.bincount(np.floor(ar / 1000.).astype(int))

plt.figure(figsize=(15,2))
plt.title("Number of Price Jumps, by 1sec intervals")
plt.xlabel("seconds")
_ = plt.plot(bc[:100], 'k-')
```

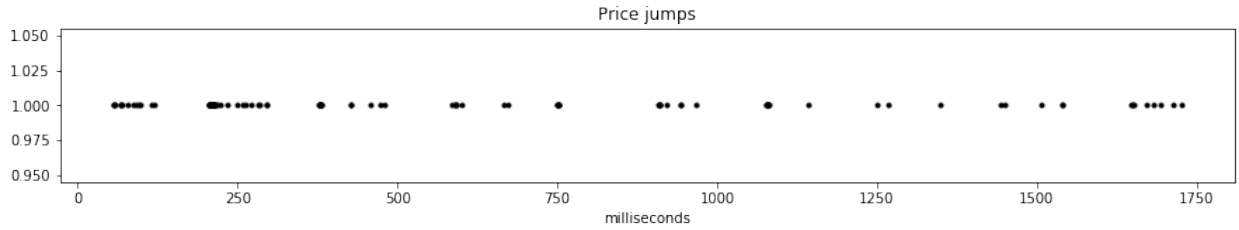


There are many reasons to prefer discrete-time models. First, data may only be collected by a real-world observer (system) in uniform grids of time – and no more granular data is available. Then, of course, more sophisticated models are of little use. Second, these models might be just enough to explain temporal relationships.

However, many real-world data are available with timestamps. That is, they correspond to discrete events in *continuous time*. The analyst's job, then, is to better explain granular temporal relationships, and answer questions like "when is the next *event* going to occur?", or "how many events do we expect in the *next* 5 minutes?". By basing the analysis on a formalism of continuous time (events can occur at any time) and discrete events (occurrences are *instantaneous*) , the answers to such questions are unlocked. Furthermore, the framing of the analysis does not depend on arbitrary discretizations of data, that may well lead to loss of valuable information.

Plotting the occurrences themselves, this is more intuitive. Each "point" in the graph is a unique data point, and an arbitrary time "grid" would lead to loss of information.

```
plt.figure(figsize=(15,2))
plt.title("Price jumps")
plt.xlabel("milliseconds")
_ = plt.plot(ar[:100], np.ones(100), 'k.')
```



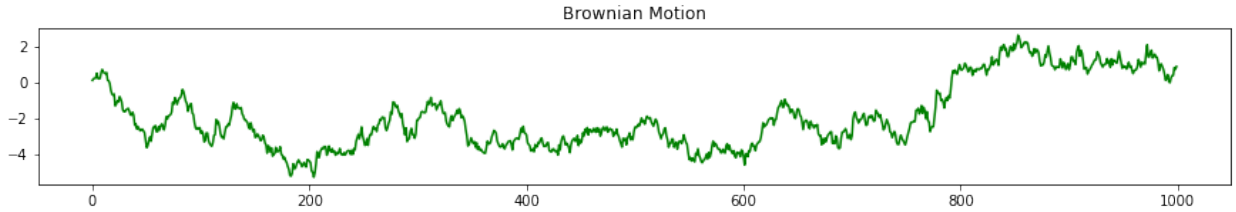
2.2 Temporal Point Processes

For dealing with the data above, we will need a few definitions. **Stochastic processes** are defined as (often *infinite*) collections of random variables. We will also equip this collection with an *index set*. For instance, formalizing a model for the "discrete-time" data above, we could write $\{X_t\}_{t=0}^{\infty}, t \in \mathbb{Z}_+$. Here X_t make up the collection while $\mathbb{Z}_+$ is the *index set*. The specific dependence (or rather, independence) structure, and other parametric assumptions of relationships among $\{X_t\}$ determine the stochastic process. Note here that a *random variate*, or a *realization* of the process is the entire trajectory determined by values taken by all X_t (often part of which we observe).

Things are slightly more interesting when the index set is $\mathbb{R}$. Interpreting the index set as time again, we have arrived at *continuous-time* processes. Each realization from the process now completely determines a *function* on domain $\mathbb{R}$.

In machine learning, a **Gaussian process** is one such example. A staple of quantitative finance, the **Wiener process** (Brownian motion) is another example.

```
ar_bm = np.cumsum(np.random.randn(1000) * 0.5**2)
plt.figure(figsize=(15,2))
plt.title("Brownian Motion")
_ = plt.plot(ar_bm, 'g-')
```

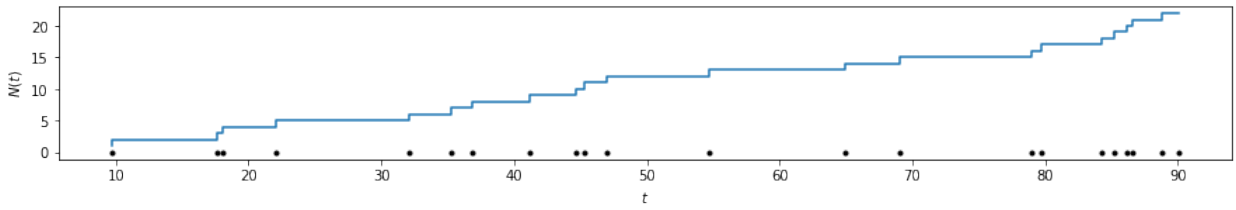


Drawing *realizations* from both Gaussian and Wiener processes (a stylized example above), we end up with *functions* $f : \mathbb{R} \to \mathbb{R}$. For our purposes, of modeling discrete events, let us restrict this family of possible functions to a special class of step functions defined on $\mathbb{R}$. Namely, we will deal with functions $N : \mathbb{R}_+ \to \mathbb{Z}_+$, which are step functions such that $s > t$ implies $N(s) \geq N(t)$. We call such processes **counting processes**.

One possible counting process realization is presented below. Intuitively, the name already suggests one interpretation close to what we are looking for. We can simply take $N(t)$ to correspond to the "number of occurrences" up to time t. This also suggests that with every counting process, we can associate a probability distribution over *points* on a timeline. This correspondence is also represented in the figure below. (A technical note here. In making this jump from counting processes to points, we will assume hereforth that no two points coincide, *almost surely*. In practice, this is rarely an issue.)

This is one way to define **temporal point processes**, a probability distribution such that each draw is a collection of points on the real line (often the "timeline"). Each "point" will correspond to an "event occurrence" in our example above, and we will use these theoretical devices to explore how "event occurrences" are dispersed throughout time.

```
ar_pp = sorted(np.random.rand(np.random.poisson(20)) * 100)
f = plt.figure(figsize=(15,2))

plt.step(ar_pp, np.cumsum(np.ones_like(ar_pp)))
plt.ylabel("$N(t)$")
plt.xlabel('$t$')
_ = plt.plot(ar_pp, np.zeros_like(ar_pp), 'k.')
```



2.3 Poisson Process

We start with the simplest temporal point process, the **Poisson process**. Poisson processes have been described as the *simplest* process, the process with *complete randomness*[1], or by Robert Gallager as "the process for which everything we could wish to be true, is true".

[1] Daley, D. J., and D. Vere-Jones. "An Introduction to the Theory of Point Processes: Volume I: Elementary Theory and Methods."

The Poisson process is characterized by complete independence. Other than the point process being *simple* (no two points coincide), the defining property of Poisson processes is as follows:

The number of occurrences on *any two disjoint intervals* is **independent**

The following property is often given in the *definition* of Poisson processes. Surprisingly, this property is in fact a consequence of the property above (and some other more technical assumptions).

The number of occurrences on an interval A **follows the Poisson distribution**,

$$N(A) \sim \mathcal{P}\wr(\xi(A))$$

Here we have let $N(A)$ denote the number of points on the interval A, which is itself a random variable of course. $\mathcal{P}\wr$ denotes the Poisson distribution. ξ is a bit more tricky. It is a *measure* on $\mathbb{R}$, such that it takes nonnegative values, satisfies $\xi(\emptyset) = 0$, and the sum of measures of disjoint sets is equal to the measure of the union of such sets. For our purposes, however, let us take

$$\xi(A) = \int_A dt\lambda(t),$$

where

$$\lambda(t) = \lim_{\Delta t \downarrow 0} \Delta t^{-1} \mathbb{E}N(t, t + \Delta t) \tag{2.1}$$

$$= \lim_{\Delta t \downarrow 0} \Delta t^{-1} \mathbb{P}\{N(t, t + \Delta t) > 0\} \tag{2.2}$$

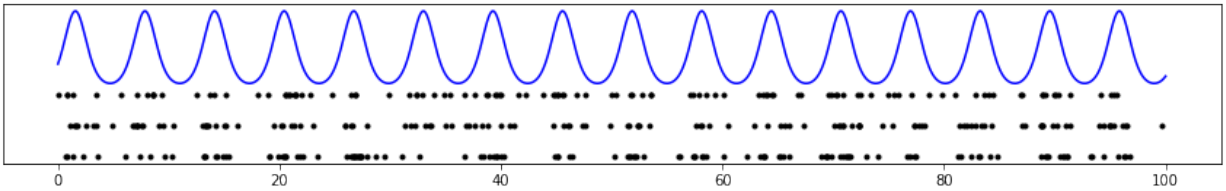
We define the function λ, the **intensity function**. For those familiar with probability theory, it should resemble the *density* function. One way to think about it is that $\lambda(t)$ defines (in the limit) the probability that there is an occurrence in the infinitesimal interval after time t. So the higher $\lambda(t)$, the higher the probability of observing points in and around t (assuming $\lambda(t)$ is smooth and nice). Let us finally note that the equality above is possible due to our assumption of simplicity – no two points can land on this infinitesimal interval.

Let's take a step back and recap.

- We define a Poisson process with a function $\lambda(t) > 0, \forall t$.
- Say we have two intervals, $A, B \subset \mathbb{R}$. The number of occurrences in these intervals will be Poisson distributed with $N(A) \sim \int_A \lambda(t)dt$, and $N(B) \sim \int_B \lambda(t)dt$.
- Most importantly, $N(A), N(B)$ are independent variables for all $A \cap B = \emptyset$.
- Higher intensity functions $\lambda(t)$, as expected, are associated with higher probabilities of event occurrences.

As a concrete example, take the following draws from a Poisson process with $\lambda(t) = \exp(\sin t))$

```
a = np.linspace(0, 100, 10000)
lt = np.exp(np.sin(a))

plt.figure(figsize=(15, 2))
plt.plot(a, lt, 'b-')
plt.yticks([])

for k in range(3):
    count = np.random.poisson(3 * 100)
    smp = [x for x in sorted(np.random.rand(count) * 100) if np.random.rand() * 3 <␣
↪np.exp(np.sin(x))]
    plt.plot(smp, np.ones_like(smp) * -1 * k, 'k.')
```

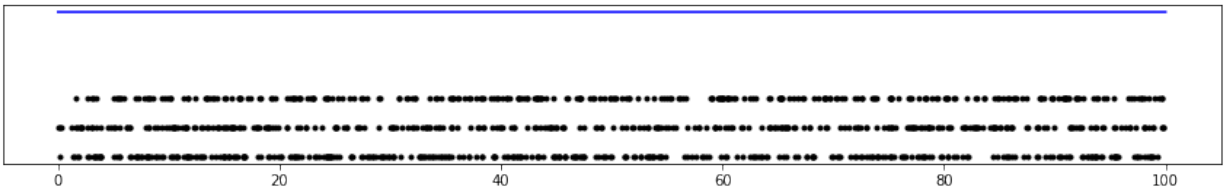


Above, the blue line represents the intensity function $\lambda(t)$, while each row of black dots is a draw from the Poisson process. Note how the dots have a higher tendency to appear near "peaks" of the intensity function.

That being said, however, the appearance of dots is completely independent. Informally, given $\lambda(t)$, each event occurs independently and is not affected by whether there are other events in its vicinity.

An important special case of the Poisson process is when the intensity function is constant, i.e. $\lambda(t) = \mu$. We call this special case a **homogeneous** Poisson process, and it is further characterized by *stationarity*. Informally, the probability that a point occurs in the vicinity of t is constant, making it equally likely for points to appear anywhere along the timeline. Concretely, samples from this process would look like (for $\lambda(t) = 3$):

```
a = np.linspace(0, 100, 10000)
lt = np.ones_like(a) * 3

plt.figure(figsize=(15, 2))
plt.plot(a, lt, 'b-')
plt.yticks([])

for k in range(3):
    count = np.random.poisson(3 * 100)
    smp = np.random.rand(count) * 100
    plt.plot(smp, np.ones_like(smp) * -1 * k, 'k.')
```



We implement homogeneous Poisson processs in `hawkeslib.PoissonProcess`.

Poisson processes underlie many applications, for example in queueing theory. There, however, people or packets arriving in a queue can reasonably be expected to obey independence. In many other applications, however, the independence assumption fails basic intuition about the domain. For instance, major financial events are known to draw (excite) others like them. Earthquakes not only occur stochastically themselves, but stochastically trigger others. In these domains, we understand, that Poisson processes lead to an oversimplification. We must work with a more expressive class of models.

2.4 Self-exciting Processes

Until now we used the real line on which we defined our point process only rather casually to represent time. The same set $\mathbb{R}$ can be used to represent distance on a fault line, or depth for example; when carrying out a "cross-sectional" analysis of earthquakes. Here, we will start assigning some meaning to time.

We are looking for ways to break the independence assumption and somehow let event occurrences depend on others. A very natural way to do this is to let the "future" (the rest of the real line deemed not observed) depend on the past. Concretely, on $\mathbb{R}$, we will let $\lambda(t)$ depend on the occurrences in $[0, t)$.

In Poisson processes, the intensity function $\lambda(t)$ was deterministic. Here, let us introduce $\lambda^*(t)$, the *conditional intensity* function. $\lambda^*(t)$ determines the probability of a point occuring in the infinitesimal interval after t, *given* the events that have *occurred* until t (the asterisk will serve as a reminder of this conditioning). In reality, $\lambda^*(t)$ is a function of t, as well as the occurrences $\{t_i | t_i < t\}$.

Let's not get into details here, but it is a fundamental result in the general theory of temporal point processes[1] that we can take $\lambda^*(t)$, and under a set of mild conditions this will lead to a well-defined point process. Furthermore, such a characterization will enable simplified calculations of likelihood and will be interpretable. See[1] chap. 7 for further details.

Processes defined as above have been called *evolutionary*, **self-modulating**, or *conditional intensity* point processes[1],[2]. In cases where a point occurrence only *increases* future $\lambda^*(t)$, another term is more appropriate: **self-exciting**.

2.5 Hawkes Processes

Hawkes processes[3] are often the first and most popular example to evolutionary processes. The (univariate) Hawkes process is defined by the conditional intensity function

$$\lambda^*(t) = \mu + \sum_{t_i < t} \varphi(t - t_i).$$

Let's take a minute to break this equation down. At any moment t, the conditional intensity function is at least $\mu > 0$, the *background intensity*. However, it also depends *linearly* on effects of events that have occurred before time t. Namely, this dependence is through a *triggering kernel* function $\varphi(.)$, a function of the *delay* $t - t_i$ between the current time and the timestamp of the previous event. Note that φ is nonnegative ($\varphi(x) \geq 0, \forall x \geq 0$ and *causal* $\varphi(x) = 0, \forall x < 0$. It is usually a monotonically decreasing function (such as exponential decay, or power-law decay).

Thinking the other way around, the function can be interpreted as follows. Each event that occurs stochastically at a time t_i adds additional intensity to the process. This added effect often *decays* throughout time (as governed by φ). In other words, every new occurrence *excites* the process, hence *self-exciting*.

The most commonly used kernel function is an exponential decay $\varphi(x) = \alpha\beta \exp(-\beta x)$. Note that this factorized form, with $\int \beta \exp(-\beta x) = 1$, leads to a convenient interpretation. $\alpha > 0$ is known as the *infectivity factor*, and defines the *average* number of new occurrences excited by any given occurrence. $\beta \exp(-\beta x)$, on the other hand is simply the exponential density function that governs the probability distribution of *delays* between events that excite each other. This is why it is also called the delay *density*.

Below is a graphical representation of $\lambda^*(t)$. Observe how the intensity is *stochastically* excited by each new arriving occurrence.

```
from hawkeslib import UnivariateExpHawkesProcess

mu, alpha, beta = .1, .2, .1
uv = UnivariateExpHawkesProcess()
uv.set_params(mu, alpha, beta)
smp = uv.sample(100)

lda_ar = [mu + np.sum(alpha * beta * np.exp(-beta * (x - smp[smp < x]))) \
          for x in np.arange(0, 100, .1)]

plt.figure(figsize=(15,2))
plt.ylabel("$\lambda^*(t)$")
plt.xlabel("$t$")
```

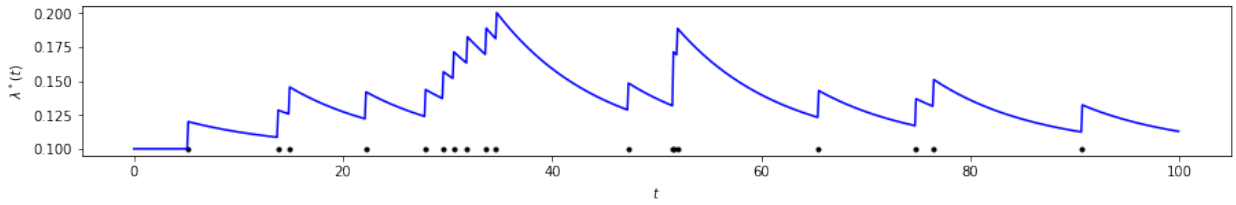
(continues on next page)

[2] Cox, David Roxbee, and Valerie Isham. Point processes. Vol. 12. CRC Press, 1980.

[3] Hawkes, Alan G. "Point spectra of some mutually exciting point processes." Journal of the Royal Statistical Society. Series B (Methodological) (1971): 438-443.

(continued from previous page)

```
plt.plot(smp, np.ones_like(smp) * .1, 'k.')
_ = plt.plot(np.arange(0, 100, .1), lda_ar, 'b-')
```



So far, we discussed "univariate" Hawkes processes. We could assume, however, that each event occurrence bears a discrete *mark* or label from a finite set. Concretely, going back to our financial example, event occurrences can belong to different types or assets. In this case, one could view the system not only as a single stochastic process, but a finite array of interacting, or *mutually-exciting* temporal point processes.

Assume observed data is now available as a set of ordered pairs $\{(t_i, c_i)\}$ where $t_i \in \mathbb{R}_+$ are the timestamps, and $c_i \in \{0, 1, \ldots, K\}$ are identifiers for which process a given occurrence belongs to. We formalize a **multivariate Hawkes process** using set of conditional intensity functions

$$\lambda_k^*(t) = \mu_k + \sum_l \sum_{t_j < t, c_j \in l} \varphi_{l,k}(t - t_j).$$

where $l, k \in \{0, 1, \ldots, K\}$ Intuitively, now each process is not only *self-excitatory* but also excited by events from other processes. Once again, it is common to take a factorized kernel of the form

$$\varphi_{l,k}(x) = A_{l,k}\theta \exp(-\theta x),$$

where now A is interpreted as the *infectivity matrix*, and $A_{l,k}$ is interpretable as the expected number of further type-k events that will be caused by events of type l.

Likelihood computation, parameter estimation and inference problems in the backdrop of Hawkes processes are not trivial, but they are beyond the scope of this short introduction. See[4],[5] for extensive surveys with a more rigorous treatment of Hawkes processes. Most implementations in this library, and their corresponding API documentation refer to the standard terminology set out in these works.

References

[4] Bacry, Emmanuel, Iacopo Mastromatteo, and Jean-François Muzy. "Hawkes processes in finance." Market Microstructure and Liquidity 1.01 (2015): 1550005.

[5] Laub, Patrick J., Thomas Taimre, and Philip K. Pollett. "Hawkes processes." arXiv preprint arXiv:1507.02822 (2015).

CHAPTER 3

Example: Modeling Seismic Activity

One field where Hawkes processes have traditionally been popular is seismology. In this example, we look at fitting a univariate Hawkes model to earthquakes.

```
%matplotlib inline
import requests
from lxml import html
from datetime import datetime as dt
import numpy as np
from matplotlib import pyplot as plt
```

Let's start by scraping some data on recent earthquakes in and around Istanbul – where this tutorial was written. The following short script uses `requests`, `lxml` and `pandas` to scrape some data from the "recent earthquakes" report maintained by Bogazici University's Kandilli Observatory, and whip it into shape for use.

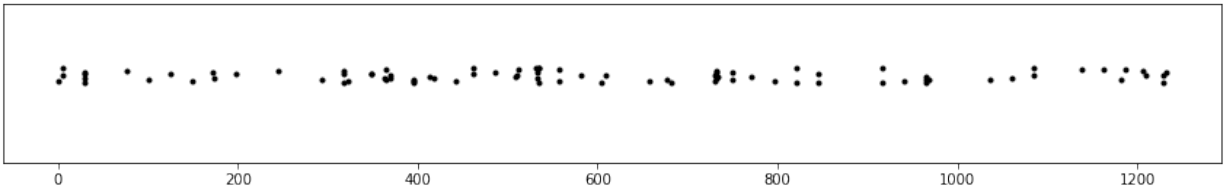
```
res = requests.get("http://www.koeri.boun.edu.tr/scripts/lst9.asp")
tx = html.fromstring(res.content).xpath("//pre/text()")[0]
lines = tx.splitlines()[7:]  # get rid of the headers

# take out timestamps and convert them to "hours since first event"
timestamps = [dt.strptime(l[:19], "%Y.%m.%d %H:%M:%S") for l in lines if "ISTANBUL"␣
↪in l]
t = np.array([(x - ts[-1]).total_seconds() / 3600 for x in timestamps])[::-1]
```

3.1 Estimating Hawkes process parameters

Hawkes processes model *self-excitement*, systems where point (or *event*) occurrences excite, or increase the probability of occurrence for, future points. Earthquakes fit this description. Indeed, they are known to trigger *after-shocks*.

Plotting occurrence times on the timeline, one thing we can expect of aftershock sequences is that they appear "clustered" in time. This is intuitive, a "main" earthquake would occur stochastically (as it would occur in a Poisson process), and the others in the cluster would follow closely after. We plot our data below to verify that this appears to be the case. (We add some "jitter" on the y axis to make viewing easier).

```
plt.figure(figsize=(15,2))
plt.ylim([-5, 5])
plt.yticks([])
_ = plt.plot(t, np.random.rand(len(t)), 'k.')
```



We now move to fitting a (univariate) Hawkes process, using the `hawkeslib.UnivariateExpHawkesProcess` class. Before we move on, let's recap the interpretation of the parameters.

- `mu` is the background intensity rate, i.e. the intensity rate for the earthquakes that occur exogenously.
- `alpha` is the infectivity factor. It can be interpreted as the number of aftershocks, in expectation, to be triggered by each earthquake.
- `theta` is the *rate* parameter of the exponential delay *density*. For example, `theta` equaling to 0.5 would mean that on average 2 hours pass between the main earthquake and the aftershock.

Let's fit the model.

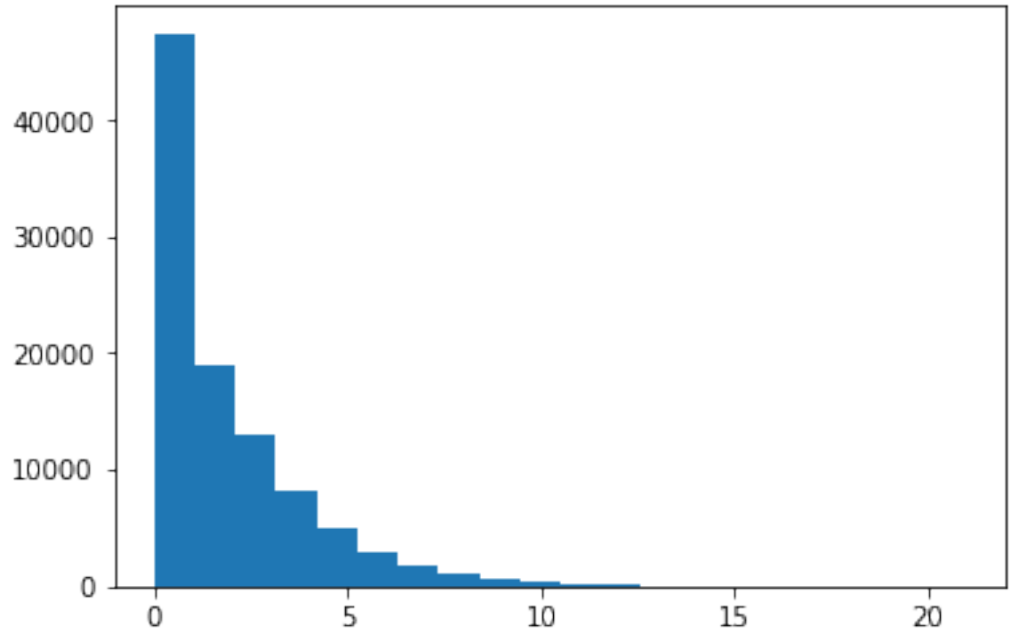
```
%%time
from hawkeslib import UnivariateExpHawkesProcess as UVHP

uv = UVHP()
uv.fit(t)
print uv.get_params()
```

```
(0.04930555306217892, 0.30548369162341404, 5.150339498582191)
CPU times: user 1.48 ms, sys: 0 ns, total: 1.48 ms
Wall time: 1.44 ms
```

Interpreting the parameters, we conclude that earthquakes occur exogenously once every ~20 hours in Istanbul (here, we call any registered seismic activity an "earthquake"). Each main shock results in 0.3 aftershocks on average, and aftershocks occur with `1 / 5.15 = 0.194` delay or 12 minutes on average.

Having fit a model, `hawkeslib` allows sampling (unconditional) from the model, as well as evaluate likelihood (e.g. for out-of-sample cross validation) for other data sets. Here, let's take a few samples from the "earthquake timeline" and use it to approximate the histogram for the number of tremors during a 24 hour time span in the city.

```
nr_shocks_sample = [len(uv.sample(24)) for x in range(100000)]
_ = plt.hist(nr_shocks_sample, bins=20)
```



3.2 Bayesian inference

Having fit the model, we now move to quantifying uncertainty in the parameter estimates. In `hawkeslib`, we do this by Bayesian inference in the univariate Hawkes model. Related functionality is implemented in `hawkeslib.BayesianUVExpHawkesProcess`.

Below, we use `hawkeslib` to sample from the posterior distribution of parameters `mu`, `alpha`, and `theta`. We then present "Bayesian credible intervals" for the parameters.

```
from hawkeslib import BayesianUVExpHawkesProcess as BUVHP

buv = BUVHP(mu_hyp=(1., 10.), alpha_hyp=(1., 1.), theta_hyp=(1., 10.))
trace = buv.sample_posterior(t, T=t[-1], n_samp=50000)
```

```
# compute the BCIs
print pm.stats.quantiles(trace["alpha"], [2.5, 97.5])
print pm.stats.quantiles(trace["theta"], [2.5, 97.5])
```

```
{2.5: 0.19881998542628126, 97.5: 0.45850986170997604}
{2.5: 2.9138895786443646, 97.5: 8.605057427283178}
```

We observe that, under small data the credible intervals around our parameters are relatively wide.

Let us end by noting that a more expressive model, one that takes into account earthquake *magnitudes*, would be required for more realistic scenarios. Traditionally, this is a *marked* Hawkes process that's known as ETAS, the Epidemic-type Aftershock Sequence Model[1].

References

[1] Ogata, Yosihiko, Ritsuko S. Matsu'ura, and Koichi Katsura. "Fast likelihood computation of epidemic type aftershock-sequence model." Geophysical research letters 20.19 (1993): 2143-2146.

CHAPTER 4

API Reference

4.1 Univariate Hawkes Processes

4.2 Multivariate Hawkes Processes

Multivariate Hawkes processes are those in which event occurrences assume discrete marks from a finite set of cardinality K. Analogously, we can think of K distinct Hawkes processes running, that not only *self-excite*, but also excite other processes (i.e. are *mutually exciting*).

4.3 Poisson Processes

For sake of completeness and comparability, we provide temporal Poisson processes (and a Bayesian variant) implementing the `PointProcess` interface, just like Hawkes processes.

The same functionality as for Hawkes; such as computing log likelihoods (or posterior potentials), maximum likelihood (or MAP) estimates of parameters, and posterior sampling are implemented. Note that due to the well-known *complete randomness* property of Poisson processes, and also the use of a conjugate prior for the Bayesian case, these methods are implemented just in a few lines of code.

CHAPTER 5

Indices and tables

- genindex
- modindex
- search