

Boltzmann Equations for the PQMSSM (v2.3)

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ABSTRACT:

1. General Formalism and Approximations

The general Boltzmann equation for the number distribution of a particle species can be written as[1] (assuming isotropy):

$$\frac{\partial F_i}{\partial t} - Hp \frac{\partial F_i}{\partial p} = C_i[F_i, F_j, p] \quad (1.1)$$

where $F_i(p)$ is the number distribution of particle i as function of momentum p , C represents a source/sink term and H is the Hubble constant:

$$H = \sqrt{\frac{8\pi}{3} \frac{\rho_T}{M_P^2}} \quad (1.2)$$

with $M_P = 1.22 \times 10^{19}$ GeV and $\rho_T = \sum_i \rho_i$. The number, energy and pressure densities are given in terms of F_i as:

$$\begin{aligned} n_i(t) &= \int \frac{dp}{2\pi^2} p^2 F_i(p) \\ \rho_i(t) &= \int \frac{dp}{2\pi^2} p^2 E_i F_i(p) \\ P_i(t) &= \frac{1}{3} \int \frac{dp}{2\pi^2} \frac{p^4}{E_i} F_i(p) \end{aligned} \quad (1.3)$$

where m_i is the mass of particle i and $E_i = \sqrt{p_i^2 + m_i^2}$. Using Eq.(1.1), we obtain the following equations for the number and energy densities:

$$\begin{aligned} \frac{dn_i}{dt} + 3Hn_i &= \int \frac{dp}{2\pi^2} p^2 C_i \\ \frac{d\rho_i}{dt} + 3H(\rho_i + P_i) &= \int \frac{dp}{2\pi^2} p^2 E_i C_i \end{aligned} \quad (1.4)$$

The collision term, C_i , for the process $i + f + \dots \leftrightarrow a + b + c + \dots$ is given by[2]:

$$\begin{aligned} C_i &= \frac{1}{E_i} \int \prod_{j,a} \frac{d^3 p_j}{2E_j(2\pi)^3} \frac{d^3 p_a}{2E_a(2\pi)^3} (2\pi)^4 \delta^4(p_i + p_j + \dots - p_a - p_b \dots) |\mathcal{M}|^2 \\ &\times [(1 \pm f_a)(1 \pm f_b) \dots f_i f_j \dots - f_a f_b \dots (1 \pm f_i)(1 \pm f_j) \dots] \end{aligned} \quad (1.5)$$

where the plus (minus) sign is for bosons (fermions). Below we always assume $f_{i,j,a,\dots} \ll 1$, so:

$$\begin{aligned} C_i &\simeq \frac{1}{E_i} \int \prod_{j,a} \frac{d^3 p_j}{2E_j(2\pi)^3} \frac{d^3 p_a}{2E_a(2\pi)^3} (2\pi)^4 \delta^4(p_i + p_j + \dots - p_a - p_b \dots) |\mathcal{M}|^2 \\ &\times [f_i f_j \dots - f_a f_b \dots] \end{aligned} \quad (1.6)$$

We will assume that C is given by:

$$C = C_{dec} + C_{prod} + C_{ann} \quad (1.7)$$

where C_{dec} contains the contributions from decays and inverse decays ($i \leftrightarrow a+b+\dots$), C_{prod} contains the contributions from decay injection and inverse decay injection ($a \leftrightarrow i+b+\dots$) and C_{ann} from annihilations with the thermal plasma ($i+i \leftrightarrow a+b$). Below we compute each term separately, under some assumptions.

1.1 Annihilation Term

The annihilation term C_{ann} for the $i + j \leftrightarrow a + b$ process is given by[1]:

$$\int \frac{dp}{2\pi^2} p^2 C_{ann} = \int d\Pi_i d\Pi_j d\Pi_a d\Pi_b (2\pi)^4 \delta^{(4)}(p_i + p_j - p_a - p_b) |M|^2 [f_a f_b - f_i f_j] \quad (1.8)$$

where $d\Pi_i = d^3p_i / ((2\pi)^3 2E_i)$. Since we are ultimately interested in Eqs.(1.4) for the number and energy densities, we will consider the following integral:

$$\int \frac{dp}{2\pi^2} p^2 C_{ann} E_i^\alpha = \int d\Pi_i d\Pi_j d\Pi_a d\Pi_b (2\pi)^4 \delta^{(4)}(p_i + p_j - p_a - p_b) |M|^2 [f_a f_b - f_i f_j] E_i^\alpha \quad (1.9)$$

where $\alpha = 0(1)$ for the number (energy) density. Here we assume that the distributions can be approximated by¹:

$$f_i \simeq \exp(-(E_i - \mu_i)/T) \quad (1.10)$$

so the annihilation term can then be written as:

$$\begin{aligned} \int \frac{dp}{2\pi^2} p^2 C_{ann} E_i^\alpha &= -(\exp((\mu_i + \mu_j)/T) - \exp((\mu_a + \mu_b)/T)) \\ &\times \int d\Pi_i d\Pi_j d\Pi_a d\Pi_b (2\pi)^4 \delta^{(4)}(p_i + p_j - p_a - p_b) |M|^2 \exp(-(E_i + E_j)/T) \times E_i^\alpha \end{aligned}$$

where above we have used conservation of energy ($E_i + E_j = E_a + E_b$). Since for the cases of interest the equilibrium distributions have zero chemical potential, we have:

$$\frac{n_i}{\bar{n}_i} = \exp(\mu_i/T) \quad (1.11)$$

so:

$$\begin{aligned} \int \frac{dp}{2\pi^2} p^2 C_{ann} E_i^\alpha &= -\left(\frac{n_i n_j}{\bar{n}_i \bar{n}_j} - \frac{n_a n_b}{\bar{n}_a \bar{n}_b}\right) \\ &\times \int d\Pi_i d\Pi_j d\Pi_a d\Pi_b (2\pi)^4 \delta^{(4)}(p_i + p_j - p_a - p_b) |M|^2 \exp(-(E_i + E_j)/T) \times E_i^\alpha \end{aligned}$$

In particular, for the process $i + i \leftrightarrow a + b$, where a and b are in thermal equilibrium ($\mu_a = \mu_b = 0$):

$$\begin{aligned} \int \frac{dp}{2\pi^2} p^2 C_{ann} E_i^\alpha &= -\left(\frac{n_i^2}{\bar{n}_i^2} - 1\right) \\ &\times \int d\Pi_i d\Pi_j d\Pi_a d\Pi_b (2\pi)^4 \delta^{(4)}(p_i + p_j - p_a - p_b) |M|^2 \exp(-(E_i + E_j)/T) \times E_i^\alpha \\ &= -(n_i^2 - \bar{n}_i^2) \langle \sigma v E_i^\alpha \rangle \end{aligned} \quad (1.12)$$

For $\alpha = 0$, the above equation is the well known contribution from thermal scatterings to the annihilation term. To estimate its value for $\alpha = 1$, we assume:

$$\langle \sigma v E \rangle \simeq \langle \sigma v \rangle \langle E_i \rangle = \langle \sigma v \rangle \frac{\rho_i}{n_i} \quad (1.13)$$

¹This approximation is only valid for particles with a thermal distribution. However, since the annihilation term is responsible for keeping the particle i in thermal equilibrium with the plasma, it is reasonable to assume a thermal distribution for i while the annihilation term is relevant.

where $\langle \rangle$ represents thermal average. Thus:

$$\int \frac{dp}{2\pi^2} p^2 C_{ann} E_i^\alpha = (\bar{n}_i^2 - n_i^2) \begin{cases} \langle \sigma v \rangle, & \text{for } \alpha = 0 \\ \langle \sigma v \rangle \frac{p_i}{n_i}, & \text{for } \alpha = 1 \end{cases} \quad (1.14)$$

1.2 Decay Term

Now we derive a simplified expression for the decay (and inverse decay) term, under approximations similar to the ones used in the last section. The decay term includes the contributions from particle decay and inverse decay[2, 3]:

$$C_{dec} \simeq \frac{1}{E_i} \int \prod_a \frac{d^3 p_a}{2E_a (2\pi)^3} (2\pi)^4 \delta^4(p_i - p_a - p_b \dots) |\mathcal{M}|^2 [f_i - f_a f_b \dots] \quad (1.15)$$

As in the case of the annihilation term, we assume that the distributions for a, b, \dots can be approximated by $f_x \simeq \exp(-(E_x - \mu_x)/T)$, so we can write:

$$f_a f_b \dots \simeq \exp\left(\frac{\mu_a + \mu_b + \dots}{T}\right) \exp(-E_i/T) = \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots} \exp(-E_i/T) = \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots} \bar{f}_i \quad (1.16)$$

where we used conservation of energy ($E_a + E_b + \dots = E_i$) and \bar{f}_i is the equilibrium distribution for the species i . Hence we can write Eq.(1.15) as:

$$\begin{aligned} C_{dec} &\simeq \left[f_i - \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots} \bar{f}_i \right] \frac{1}{E_i} \int \prod_a \frac{d^3 p_a}{2E_a (2\pi)^3} (2\pi)^4 \delta^4(p_i - p_a - p_b \dots) |\mathcal{M}|^2 \\ &= \mathcal{B}_{ab\dots} \frac{\Gamma_i m_i}{E_i} \left[f_i - \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots} \bar{f}_i \right] \end{aligned} \quad (1.17)$$

where Γ_i is the width for i and $\mathcal{B}_{ab\dots} \equiv BR(i \rightarrow a + b + \dots)$

Once again we consider the integral:

$$\begin{aligned} \int \frac{dp}{2\pi^2} p^2 C_{dec}(p) E_i^\alpha &= -\Gamma_i \int \frac{dp}{2\pi^2} p^2 \frac{m_i}{E_i} f_i E_i^\alpha \\ &+ \sum_{i \text{ decays}} \mathcal{B}_{ab\dots} \Gamma_i \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots} \int \frac{dp}{2\pi^2} p^2 \frac{m_i}{E_i} \bar{f}_i E_i^\alpha \end{aligned} \quad (1.18)$$

where we have included the sum over all decay channels and $\alpha = 0(1)$ for the contribution to the number (energy) density equation. Note that both integrals are identical, except for the replacement $f_i \rightarrow \bar{f}_i$. The first integral in Eq.(1.18) gives:

$$-\Gamma_i \int \frac{dp}{2\pi^2} p^2 \frac{m_i}{E_i} f_i(p) E_i^\alpha = \begin{cases} -\Gamma_i m_i n_i \langle \frac{1}{E_i} \rangle, & \text{for } \alpha = 0 \\ -\Gamma_i m_i n_i, & \text{for } \alpha = 1 \end{cases} \quad (1.19)$$

where

$$\langle \frac{1}{E_i} \rangle \equiv \frac{1}{n_i} \int \frac{dp}{2\pi^2} p^2 \frac{1}{E_i} f_i(p) \quad (1.20)$$

Hence we can write Eq.(1.18) as:

$$\int \frac{dp}{2\pi^2} p^2 C_{dec}(p) E_i^\alpha = -\Gamma_i m_i \begin{cases} n_i \langle \frac{1}{E_i} \rangle - \bar{n}_i \langle \frac{1}{E_i} \rangle_{eq} \sum \mathcal{B}_{ab\dots} \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots}, & \text{for } \alpha = 0 \\ n_i - \bar{n}_i \sum \mathcal{B}_{ab\dots} \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots}, & \text{for } \alpha = 1 \end{cases} \quad (1.21)$$

For the non-equilibrium average we assume:

$$\langle \frac{1}{E_i} \rangle \simeq \frac{1}{\langle E_i \rangle} = \frac{n_i}{\rho_i} \quad (1.22)$$

which is exact in the non-relativistic limit, but it is only an approximation for the relativistic case. Although we can compute the equilibrium average ($\langle \frac{1}{E_i} \rangle_{eq}$) explicitly, in order to have an exact cancellation between the decay and inverse decay terms when i , a and b are all in equilibrium, we take:

$$\langle \frac{1}{E_i} \rangle_{eq} \simeq \langle \frac{1}{E_i} \rangle = \frac{n_i}{\rho_i} \quad (1.23)$$

With the above approximations we finally obtain:

$$\int \frac{dp}{2\pi^2} p^2 C_{dec}(p) E_i^\alpha = -\Gamma_i m_i \begin{cases} \frac{n_i}{\rho_i} \left(n_i - \bar{n}_i \sum \mathcal{B}_{ab\dots} \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots} \right), & \text{for } \alpha = 0 \\ n_i - \bar{n}_i \sum \mathcal{B}_{ab\dots} \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots}, & \text{for } \alpha = 1 \end{cases} \quad (1.24)$$

where $\mathcal{B}_{ab\dots} \equiv BR(i \rightarrow a + b + \dots)$.

1.3 Production Term

The decay and inverse decay of other particles ($a \rightarrow i + b + \dots$) can also affect the species i . The contribution from these terms we label C_{prod} , which is given by[2]:

$$C_{prod} \simeq \frac{1}{E_i} \int \frac{d^3 p_a}{2E_a (2\pi)^3} \prod_b \frac{d^3 p_b}{2E_b (2\pi)^3} (2\pi)^4 \delta^4(p_a - p_i - p_b \dots) |\mathcal{M}|^2 [f_a - f_i f_b \dots] \quad (1.25)$$

Using the same approximations of the previous section, we write:

$$f_i f_b \dots \simeq \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} e^{-E_a/T} = \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} \bar{f}_a \quad (1.26)$$

Hence:

$$C_{prod} = \frac{1}{E_i} \int \frac{d^3 p_a}{2E_a (2\pi)^3} \prod_b \frac{d^3 p_b}{2E_b (2\pi)^3} (2\pi)^4 \delta^4(p_a - p_i - p_b \dots) |\mathcal{M}|^2 \left(f_a - \bar{f}_a \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} \right) \quad (1.27)$$

and

$$\begin{aligned} \int \frac{dp}{2\pi^2} p^2 C_{prod}(p) E_i^\alpha &= \int \frac{d^3 p_a}{E_a (2\pi)^3} \left(f_a - \bar{f}_a \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} \right) \\ &\times \frac{d^3 p E_i^\alpha}{2E_i (2\pi)^3} \prod_b \frac{d^3 p_b}{2E_b (2\pi)^3} (2\pi)^4 \delta^4(p_a - p_i - p_b \dots) |\mathcal{M}|^2 \end{aligned} \quad (1.28)$$

with $\alpha = 0(1)$ for the contribution to the number (energy) density equation. For $\alpha = 0$ we obtain:

$$\begin{aligned} \int \frac{dp}{2\pi^2} p^2 C_{prod}(p) &= \Gamma_a \mathcal{B}_i m_a \int \frac{d^3 p_a}{E_a (2\pi)^3} \left(f_a - \bar{f}_a \sum_b \frac{\mathcal{B}_{ib\dots}}{\mathcal{B}_i} \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} \right) \\ &= \Gamma_a \mathcal{B}_i m_a \frac{n_a}{\rho_a} \left(n_a - \bar{n}_a \sum_b \frac{\mathcal{B}_{ib\dots}}{\mathcal{B}_i} \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} \right) \end{aligned} \quad (1.29)$$

where $\mathcal{B}_{ib\dots} \equiv BR(a \rightarrow i + b + \dots)$, $\mathcal{B}_i = \sum_b \mathcal{B}_{ib\dots}$ and we have once again assumed $\langle 1/E_a \rangle \simeq \langle 1/E_a \rangle_{eq} \simeq n_a/\rho_a$.

For $\alpha = 1$, the integral in Eq.(1.28) does not take a simple form. In order to compute it, we assume:

$$E_i \simeq \frac{E_a}{2} \quad (1.30)$$

The above expression is only exact for 2-body decays and $m_a \gg m_i, m_b$. For the remaining cases, it is only an estimate.

$$\begin{aligned} \int \frac{dp}{2\pi^2} p^2 C_{prod}(p) E_i &\simeq \Gamma_a \mathcal{B}_i \frac{m_a}{2} \int \frac{d^3 p_a}{(2\pi)^3} \left(f_a - \bar{f}_a \sum_b \frac{\mathcal{B}_{ib\dots}}{\mathcal{B}_i} \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} \right) \\ &= \Gamma_a \mathcal{B}_i \frac{m_a}{2} \left(n_a - \bar{n}_a \sum_b \frac{\mathcal{B}_{ib\dots}}{\mathcal{B}_i} \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} \right) \end{aligned} \quad (1.31)$$

Combining the results for $\alpha = 0$ and 1, we have:

$$\int \frac{dp}{2\pi^2} p^2 C_{prod}(p) E_i^\alpha = \Gamma_a \mathcal{B}_i m_a \left(n_a - \bar{n}_a \sum_b \frac{\mathcal{B}_{ib\dots}}{\mathcal{B}_i} \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} \right) \begin{cases} \frac{n_a}{\rho_a}, & \text{for } \alpha = 0 \\ \frac{1}{2}, & \text{for } \alpha = 1 \end{cases} \quad (1.32)$$

1.4 Number and Energy Density Equations

Using the results of Eqs.(1.14), (1.24) and (1.32) in the Boltzmann equations for n_i and ρ_i (Eq.(1.4)), we obtain:

$$\begin{aligned} \frac{dn_i}{dt} + 3Hn_i &= (\bar{n}_i^2 - n_i^2) \langle \sigma v \rangle - \Gamma_i m_i \frac{n_i}{\rho_i} \left(n_i - \bar{n}_i \sum_{i \rightarrow \dots} \mathcal{B}_{ab\dots} \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots} \right) \\ &+ \sum_a \Gamma_a \mathcal{B}_i m_a \frac{n_a}{\rho_a} \left(n_a - \bar{n}_a \sum_{a \rightarrow i\dots} \frac{\mathcal{B}_{ib\dots}}{\mathcal{B}_i} \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} \right) + C_i(T) \end{aligned} \quad (1.33)$$

$$\begin{aligned} \frac{d\rho_i}{dt} + 3H(\rho_i + P_i) &= (\bar{n}_i^2 - n_i^2) \langle \sigma v \rangle \frac{\rho_i}{n_i} - \Gamma_i m_i \left(n_i - \bar{n}_i \sum_{i \rightarrow \dots} \mathcal{B}_{ab\dots} \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots} \right) \\ &+ \sum_a \Gamma_a \mathcal{B}_i \frac{m_a}{2} \left(n_a - \bar{n}_a \sum_{a \rightarrow i\dots} \frac{\mathcal{B}_{ib\dots}}{\mathcal{B}_i} \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} \right) + \tilde{C}_i(T) \frac{\rho_i}{n_i} \end{aligned} \quad (1.34)$$

where $\mathcal{B}_{ab\dots} = BR(i \rightarrow a + b + \dots)$, $\mathcal{B}_{ib\dots} = BR(a \rightarrow i + b + \dots)$, $\mathcal{B}_i = \sum_b \mathcal{B}_{ib\dots}$ and we have included an extra term (C_i and \tilde{C}_i) to allow for other possible sources for the number and energy densities. For simplicity we assume $C_i = \tilde{C}_i$ from now on.

It is also convenient to use the above results to obtain a simpler equation for ρ_i/n_i :

$$\frac{d\rho_i/n_i}{dt} \equiv \frac{dR_i}{dt} = -3H \frac{P_i}{n_i} + \sum_a \mathcal{B}_i \frac{\Gamma_a m_a}{n_i} \left(\frac{1}{2} - \frac{n_a}{\rho_a} \frac{\rho_i}{n_i} \right) \left(n_a - \bar{n}_a \sum_{a \rightarrow i\dots} \frac{\mathcal{B}_{ib\dots}}{\mathcal{B}_i} \frac{n_i n_b \dots}{\bar{n}_i \bar{n}_b \dots} \right) \quad (1.35)$$

Besides the above equations, it is useful to consider the evolution equation for entropy:

$$dS \equiv \frac{dQ^{dec}}{T} \quad (1.36)$$

where dQ^{dec} is the net energy injected from decays. With the above definition we have:

$$\begin{aligned}\dot{S} &= \frac{1}{T} \sum_i BR(i, X) \frac{d(R^3 \rho_i)^{dec}}{dt} \\ \Rightarrow \dot{S} &= \frac{R^3}{T} \sum_i BR(i, X) \Gamma_i m_i \left(n_i - \bar{n}_i \sum_{i \rightarrow \dots} \mathcal{B}_{ab\dots} \frac{n_a n_b \dots}{\bar{n}_a \bar{n}_b \dots} \right)\end{aligned}\quad (1.37)$$

where R is the scale factor and $BR(i, X)$ is the fraction of energy injected in the thermal bath from i decays.

The above expressions can be written in a more compact form if we define the following "effective thermal densities" and "effective BR":

$$\begin{aligned}\mathcal{N}_X^{th} &\equiv \bar{n}_X \sum_{X \rightarrow \dots} BR(X \rightarrow 1 + 2 + \dots) \prod_k \frac{n_k}{\bar{n}_k} \\ \mathcal{N}_{XY}^{th} &\equiv \frac{\bar{n}_X}{\mathcal{B}_{XY}^{eff}} \sum_{X \rightarrow Y + \dots} g_Y BR(X \rightarrow g_Y Y + 1 + \dots) \left(\frac{n_Y}{\bar{n}_Y} \right)^{g_Y} \prod_k \frac{n_k}{\bar{n}_k} \\ \mathcal{B}_{XY}^{eff} &\equiv \sum_{X \rightarrow Y + \dots} g_Y BR(X \rightarrow g_Y Y + 1 + \dots)\end{aligned}$$

where g_Y is the Y multiplicity in the final state of X decays. In addition, defining:

$$x = \ln(R/R_0), \quad N_i = \ln(n_i/s_0), \quad \text{and} \quad N_S = \ln(S/S_0) \quad (1.38)$$

we can write Eqs.(1.37), (1.33) and (1.35) as:

$$N'_S = \frac{e^{(3x-N_S)}}{HT} \sum_i BR(i, X) \Gamma_i m_i \left(n_i - \mathcal{N}_i^{th} \right) \quad (1.39)$$

$$\begin{aligned}N'_i &= -3 + \frac{\langle \sigma v \rangle_i}{H} n_i \left[\left(\frac{\bar{n}_i}{n_i} \right)^2 - 1 \right] - \frac{\Gamma_i m_i}{H R_i} \left(1 - \frac{\mathcal{N}_i^{th}}{n_i} \right) \\ &\quad + \sum_a \mathcal{B}_{ai}^{eff} \frac{\Gamma_a m_a}{H R_a} \left(\frac{n_a}{n_i} - \frac{\mathcal{N}_{ai}^{th}}{n_i} \right)\end{aligned}\quad (1.40)$$

$$R'_i = -3 \frac{P_i}{n_i} + \sum_a \mathcal{B}_{ai}^{eff} \frac{\Gamma_a}{H} m_a \left(\frac{1}{2} - \frac{R_i}{R_a} \right) \left(\frac{n_a}{n_i} - \frac{\mathcal{N}_{ai}^{th}}{n_i} \right) \quad (1.41)$$

where $' = d/dx$.

The above equation for N_i also applies for coherent oscillating fields, if we define:

$$N_i = \ln(n_i/s_0), \quad \text{and} \quad n_i \equiv \rho_i/m_i \quad (1.42)$$

so

$$\begin{aligned}N'_i &= -3 - \frac{\Gamma_i}{H} \\ R'_i &= 0\end{aligned}\quad (1.43)$$

where we assume that the coherent oscillating component does not couple to any of the other fields.

Collecting Eqs.(1.39)-(1.41) and (1.43) we have a closed set of first order differential equations:

- Entropy:

$$N'_S = \frac{e^{(3x-N_S)}}{HT} \sum_i BR(i, X) \Gamma_i m_i \left(n_i - \mathcal{N}_i^{th} \right) \quad (1.44)$$

- Thermal fields:

$$\begin{aligned} N'_i &= -3 + \frac{\langle \sigma v \rangle_i}{H} n_i \left[\left(\frac{\bar{n}_i}{n_i} \right)^2 - 1 \right] - \frac{\Gamma_i m_i}{H R_i} \left(1 - \frac{\mathcal{N}_i^{th}}{n_i} \right) + \sum_a \mathcal{B}_{ai}^{eff} \frac{\Gamma_a m_a}{H R_a} \left(\frac{n_a}{n_i} - \frac{\mathcal{N}_{ai}^{th}}{n_i} \right) \\ R'_i &= -3 \frac{P_i}{n_i} + \sum_a \mathcal{B}_{ai}^{eff} \frac{\Gamma_a}{H} m_a \left(\frac{1}{2} - \frac{R_i}{R_a} \right) \left(\frac{n_a}{n_i} - \frac{\mathcal{N}_{ai}^{th}}{n_i} \right) \end{aligned} \quad (1.45)$$

- Coherent Oscillating fields:

$$\begin{aligned} N'_i &= -3 - \frac{\Gamma_i}{H} \\ R'_i &= 0 \end{aligned} \quad (1.46)$$

As seen above, the equation for $R_i = \rho_i/n_i$ depends on P_i/n_i . A proper evaluation of this quantity requires knowledge of the distribution $F_i(p, t)$. However, for relativistic (or massless) particles we have $P_i = \rho_i/3$, as seen from Eq.(1.3), while for particles at rest we have $P_i = 0$. Hence $F_i(p, t)$ is only required to evaluate the relativistic/non-relativistic transition, which corresponds to a relatively small part of the evolution history of particle i . Nonetheless, to model this transition we approximate F_i by a thermal distribution and take $T_i, \mu_i \ll m_i$, where T_i is the temperature of the particle (which can be different from the thermal bath's). Under these approximations we have:

$$\begin{aligned} \frac{P_i}{n_i} &= T_i \\ \frac{\rho_i}{n_i} &= T_i \left[\frac{K_1(m_i/T_i)}{K_2(m_i/T_i)} \frac{m_i}{T_i} + 3 \right] \end{aligned} \quad (1.47)$$

where $K_{1,2}$ are the modified Bessel functions. In particular, if $m_i/T_i \gg 1$:

$$\frac{\rho_i}{n_i} \simeq T_i \left[\frac{3}{2} + \frac{m_i}{T_i} + 3 \right] \Rightarrow \frac{P_i}{n_i} = T_i = \frac{2m_i}{3} \left(\frac{R_i}{m_i} - 1 \right) \quad (1.48)$$

As shown above, for a given value of $R_i = \rho_i/n_i$, Eq.(1.47) can be inverted to compute T_i ($= P_i/n_i$):

$$\frac{P_i}{n_i} = T_i(R_i) \quad (1.49)$$

Since we are interested in the non-relativistic/relativistic transition, we can expand the above expression around $R_i/m_i = 1$, so P_i/n_i can be written as:

$$\frac{P_i}{n_i} = \frac{2m_i}{3} \left(\frac{R_i}{m_i} - 1 \right) + m_i \sum_{n>1} a_n \left(\frac{R_i}{m_i} - 1 \right)^n \quad (1.50)$$

where the coefficients a_n can be numerically computed from Eq.(1.47). The above approximation should be valid for $m_i/T_i \gtrsim 1$ (or $R_i \gtrsim m_i$). On the other hand, for $m_i/T_i \ll 1$

(or $R_i \gg m_i$), we have the relativistic regime, with $P_i/n_i = R_i/3$. Therefore we can approximate the P_i/n_i function for all values of R_i by:

$$\frac{P_i}{n_i} = \begin{cases} \frac{2m_i}{3} \left(\frac{R_i}{m_i} - 1 \right) + m_i \sum_{n>1} a_n \left(\frac{R_i}{m_i} - 1 \right)^n, & \text{for } R_i < \tilde{R} \\ \frac{R_i}{3}, & \text{for } R_i > \tilde{R} \end{cases} \quad (1.51)$$

where the coefficients a_n are given by the numerical fit of Eq.(1.47) and \tilde{R} is given by the matching of the two solutions.

Finally, to solve Eqs.(1.44)-(1.46) we need to compute H according to Eq.(1.2), which requires knowledge of the energy densities for all particles (ρ_i) and for the thermal bath (ρ_R). The former are directly obtained from N_i and R_i , while the latter can be computed from N_S :

$$T = \left(\frac{g_{*S}(T_R)}{g_{*S}(T)} \right)^{1/3} T_R \exp[N_S/3 - x] \Rightarrow \rho_R = \frac{\pi^2}{30} g_*(T) T^4 \quad (1.52)$$

Eqs.(1.44)-(1.46), with the auxiliary equations for H (Eq.(1.2)) and P_i/n_i (Eq.(1.51)) form a set of closed equations, which can be solved once the initial conditions for the number density (n_i), energy density (ρ_i) and entropy (S) are given. For thermal fluids we assume:

$$n_i(T_R) = \begin{cases} 0 & , \text{ if } \langle \sigma v \rangle_i \bar{n}_i / H|_{T=T_R} < 10 \\ \bar{n}_i(T_R) & , \text{ if } \langle \sigma v \rangle_i \bar{n}_i / H|_{T=T_R} > 10 \end{cases} \quad (1.53)$$

$$\frac{\rho_i}{n_i}(T_R) = \frac{\bar{\rho}_i}{\bar{n}_i}(T_R) \quad (1.54)$$

where $\bar{\rho}_i$ is the equilibrium energy density (with zero chemical potential) for the particle i . While for coherent oscillating fluids the initial condition is set at the beginning of oscillations:

$$n_i(T_i^{osc}) = \frac{\rho_i^0}{m_i(T_i^{osc})} \quad (1.55)$$

$$\frac{\rho_i}{n_i}(T_i^{osc}) = m_i \quad (1.56)$$

where T_i^{osc} is the oscillation temperature, given by $3H(T_i^{osc}) = m_i(T_i^{osc})$ and ρ_i^0 the initial energy density for oscillations.

Finally, the initial condition for the entropy S is trivially obtained, once we assume a radiation dominated universe at $T = T_R$:

$$S(T_R) = \frac{2\pi^2}{45} g_*(T_R) T_R^3 R_0^3 \quad (1.57)$$

2. Code

Here we describe how the above formalism is implemented in a numerical code for solving the coupled Boltzmann equations. In Sec.2.1 we describe how the input for specific models should be defined. Then, in Sec.2.2 we outline the procedure used to solve the Boltzmann equations and to treat some of the discrete transitions required by the formalism described above. Finally, in Sec.2.3 we describe what is the output of the code and how it can be controlled by the user.

2.1 Input

In order to solve the Boltzmann equations for a particular model, the user has to provide the SUBROUTINE INPUTBOLTZ(T), which, for a given (thermal bath) temperature T , fills the COMMON BLOCK:

```
COMMON/INBOLTZ/BR(NP,NP),DEGF(NP),MASS(NP),GAM(NP),  
SIGV(NP),C(NP),COHOSC(NP),TRH,NCOMPS,LABEL(NP)
```

where $NP = 20$ and

- $NCOMPS (\leq 10)$ = the number of particles (the first component must be radiation)
- $MASS(i)$ = mass for particle i (can be temperature dependent, as in the axion case)
- $DEGF(i)$ = +-number of degrees of freedom for particle i . A plus sign should be used for bosons, while a minus should be used for fermions, i.e. $DEGF=-2$ for neutralinos and $DEGF=1$ for axions. The value for the $i=1$ component (radiation) is never used, since the number of degrees of freedom in this case is given by the function $GSTAR(T)$.
- $GAM(i)$ = decay width for particle i , in its rest frame.
- $BR(i,j)$ = branching ratio for the decay $i \rightarrow j + X$, including the multiplicity factor, if the i particle decays into multiple j 's.
- $BR(i,1)$ = fraction of energy per i particle injected in the radiation fluid.
- $SIGV(i)$ = thermal averaged cross-section for the annihilation of i particles, as defined in the previous section.
- $C(i)$ = additional source term for particle i , as defined in the previous section.
- $COHOSC(i)$ = initial energy density for coherent oscillating particles. Must be zero for thermal (non-oscillating) components.
- TRH = re-heat temperature.
- $LABEL(i)$ = label for particle i (optional)

2.2 Main Code

Once the INPUTBOLTZ subroutine is provided, the user can compute the solution for the Boltzmann equations from $T=TRH$ to $T=TF$, calling:

```
CALL INPUTBOLTZ(TRH) ! (initialization)  
CALL EQSBOLTZ(TF,IOUT) ! (compute solution)
```

where, if $\text{IOUT} > 0$, the scale factor (R) and energy densities as a function of T are written to $\text{UNIT}=\text{IOUT}$. If $\text{TF} = 0$, the evolution proceeds until all unstable particles have decayed and/or all coherent oscillating components have started to oscillate. Before calling `EQSBOLTZ`, the user must define the parameters which regulate the precision of the procedure, given by the `BLOCK DEPARS`:

`COMMON/DEPARS/EPS,DX0,STEP,IERROR`

where EPS is the relative precision for the solution $N_i(TF)$, DX0 is the x interval for printing the solutions in IOUT and the maximum Δx step and STEP is the initial Δx step for the evolution. Failure to solve the equations (most likely due to numerical instabilities) is indicated by $\text{IERROR} < 0$.

Specific components can be turned off using the `COMMON BLOCK`:

`COMMON/SWITCHES/TURNOFF(NP)`

If $\text{TURNOFF}(I)=\text{TRUE.}$, the i -component will not be included in the evolution of the Boltzmann equations.

The `EQSBOLTZ` is the main subroutine used to solve the equations, once the appropriate input has been defined. Its main steps are:

1. Set initial conditions at $T = \text{TRH}$: check which thermal particles are coupled/decoupled to the thermal bath and if coherent oscillating fluids are already oscillating at $T=\text{TRH}$. Then it sets the initial number densities and temperatures for each component, as defined in the previous section. Set $\text{X1}=1$ and $\text{X2}=\text{X1}+\text{DX0}$.
2. Solve the equations between X1 and X2 .
3. Check if a particle has decayed. If the particle i satisfies

$$\Gamma_i/H > 100 \quad \text{and} \quad \min_{j \neq i}(\rho_i/\rho_j) < 10^{-3} \quad (2.1)$$

the particle is neglected from here on. The decay temperature (T_D) is defined by the sudden decay approximation: $\Gamma_i/\gamma_i = H(T_D)$, where γ_i is the boost factor ($\gamma_i \equiv \langle E_i \rangle / m_i$). Although this temperature is printed out in the output, it is never used in the code.

4. Check if a particle has decoupled from the thermal bath or started to oscillate in the interval $(\text{X1}, \text{X2})$. Decoupling is assumed if $\langle \sigma v \rangle_i \bar{n}_i < H/10$, which also defines the freeze-out temperature. The oscillation temperature is given by $3H(T_{\text{osc}}) = m_i(T_{\text{osc}})$ and defines the beginning of evolution for the oscillating components.
5. If a component has started to oscillate or if it has decoupled, loop over this interval with smaller steps until the decoupling or oscillation temperature converges ($\Delta T_i/T_i < 0.1$).
6. Write temperature, scale factor and energy densities to IOUT , if $\text{IOUT} > 0$.
7. Set $\text{X1}=\text{X2}$ and $\text{X2}=\text{X1}+\text{DX0}$ and return to point 2. until $T < \text{TF}$ (or all unstable particles have decayed and all oscillating fluids have oscillated, if $\text{TF} \leq 0$).

2.3 Output

The standard information printed after solving the Boltzmann equations gives:

- Freeze-out temperatures (T_{fr}) for each thermal component. As mentioned in the last section, T_{fr} is given by the decoupling condition: $\langle\sigma v\rangle_i \bar{n}_i = H/10$. Since the decoupling is a continuous process, T_{fr} is just an estimate for the decoupling temperature.
- Decay temperatures (T_D) for each thermal component. Once again the decay process is continuous and T_D given in the print out is estimated by the sudden decay approximation ($\Gamma_i/\gamma_i = H(T_D)$).
- Oscillation temperature (T_{osc}).
- Entropy ratio (S/S_0). In case of entropy injection from decays of unstable particles, $S/S_0 > 1$.
- Relic densities ($\Omega_i h^2$) at $T_0 = 2.725$ K. In order to consistently compute the relic densities today, we evolve $R_i = \rho_i/n_i$ from TF to T_0 assuming a trivial universe expansion:

$$R'_i = -3 \frac{P_i}{n_i} \quad (2.2)$$

Note that the result obtained above is insensitive to H , so it does not matter if there is a transition from a radiation dominated to a matter dominated (or dark energy dominated) universe between TF and T_0 . Once $R_i(T_0)$ is obtained, the relic density is given by:

$$\Omega_i h^2 = n_i(TF) \times \frac{g_{*S}(T_0) T_0^3}{g_{*S}(TF) TF^3} \times \frac{R_i(T_0)}{\rho_c/h^2} \quad (2.3)$$

- Relic densities before decay. It may be relevant to compute the relic densities of an unstable particle as it would be given if it had not decayed. In particular, this value can be used to impose BBN bounds on the decays. This quantity is computed after the particle becomes non-relativistic and well before the decay starts ($\Gamma_i/H(T) = 1/10$) and is given by:

$$\tilde{\Omega}_i h^2 = \frac{\rho_i(T)}{s(T)} \times \frac{s(T_0)}{\rho_c/h^2} \quad (2.4)$$

Note that the above expression assumes a radiation dominated universe from T to T_0 and should be used with caution.

- Effective number of (new) neutrinos (ΔN_{eff}). Since neutrinos are still coupled for $T > 1$ MeV, this quantity is only computed below this temperature. ΔN_{eff} is given by:

$$\Delta N_{eff}(T) = \frac{\rho_{DR}(T)}{\rho_\nu} \quad (2.5)$$

where ρ_{DR} is the total energy density of relativistic particles (excluding radiation and neutrinos) and ρ_ν is the energy density of neutrinos after they freeze-out:

$$\rho_{DR} = \sum_{R_i/m_i > 2} \rho_i \text{ and } \rho_\nu = \frac{\pi^2}{15} \frac{7}{8} \left(\frac{4}{11} \right)^{4/3} T^4 \quad (2.6)$$

Note that ΔN_{eff} is in general a function of temperature, since ρ_{DR} will decrease if massive particles become non-relativistic below 1 MeV.

Furthermore, if $IOUT > 0$, the scale factor (R), the energy densities and ΔN_{eff} are printed as a function of T in `UNIT=IOUT`. Also, the following quantities are stored in `COMMON BLOCKS`:

- Final relic densities and entropy ratio:

`COMMON/OUTPUT/OMEGA(NP),RS`

- Decoupling, oscillation and decay temperatures:

`COMMON/TEMPS/TDEC(NP),TOSC(NP),TDCAY(NP)`

- Relic density of unstable particles before decay ($\tilde{\Omega}h^2$), temperature at the end of entropy injection (if any) and effective number of new neutrinos (ΔN_{eff}) *at the final temperature TF* :

`COMMON/BBNINFO/UMEGA(NP),TSTAB,DNeff`

3. PQMSSM

In order to apply the above formalism to the PQMSSM we need to define:

- Masses: $m_{\tilde{Z}_1}, m_{\tilde{a}}, m_s, m_a(T), m_{\tilde{G}}$
- Decay Width: Γ_i , with $i = \tilde{Z}_1, \tilde{a}, s, \tilde{G}$,
- Branching Ratios: $BR(i, j)$ and $BR(i, 1)$, with $i, j = \tilde{Z}_1, \tilde{a}, s, \tilde{G}$,
- Annihilation cross-sections: $\langle\sigma v\rangle_i$, with $i = a, \tilde{Z}_1, \tilde{a}, s, \tilde{G}$,
- Additional Source terms: C_i , with $i = a, \tilde{Z}_1, \tilde{a}, s, \tilde{G}$,
- Initial energy density for coherent oscillating fields: ρ_i^0 , with $i = a, s$
- and remaining SUSY spectrum (for computation of g_* and gravitino/axino decays)

Below we describe how the quantities $\langle\sigma v\rangle_i, \Gamma_i, BR(i, j), C_i$ and ρ_i^0 are computed.

3.1 $\langle\sigma v\rangle$

The annihilation cross-section for the neutralino component is computed in the SUBROUTINE `ZSIG(T,MZ)`, where T is the temperature and MZ is the neutralino mass. For efficiency purposes the calculation of $\langle\sigma v\rangle_{\tilde{Z}_1}$ can be controlled through the `COMMON BLOCK`:

`COMMON/INSIGMA/INOMGZ,INMZ,INFLAG,INDATA`

The main options are set by INFLAG:

- If INFLAG=166, the subroutine returns a constant value for $\langle\sigma v\rangle_{\tilde{Z}_1}$, given by:

$$\langle\sigma v\rangle_{\tilde{Z}_1} = 1.7 \times 10^{-10} / \Omega_{\tilde{Z}_1} h^2 \quad (3.1)$$

where the value for $\Omega_{\tilde{Z}_1} h^2$ should be set in INOMGZ.

- If INFLAG=266, then
 - If INDATA < 0: generate file with $\langle\sigma v\rangle_{\tilde{Z}_1}$ values for $3 \times 10^{-5} < T/m_{\tilde{Z}_1} < 2$ in UNIT=ABS(INDATA) for future extrapolation and set INDATA = ABS(INDATA). The number of points and specific values of T are chosen as to properly describe the shape of $\langle\sigma v\rangle_{\tilde{Z}_1}$. The file must be open by the main program.
 - If INDATA > 0: use values in UNIT=INDATA for a linear extrapolation in $\log(T/m_{\tilde{Z}_1})$
- If INFLAG \neq 166 and 266: compute $\langle\sigma v\rangle_{\tilde{Z}_1}(T)$ (full integration). Due to convergence issues, $\langle\sigma v\rangle_{\tilde{Z}_1}(T > m_{\tilde{Z}_1}/2) \equiv \langle\sigma v\rangle_{\tilde{Z}_1}(T = m_{\tilde{Z}_1}/2)$ and $\langle\sigma v\rangle_{\tilde{Z}_1}(T < m_{\tilde{Z}_1}/5 \times 10^{-5}) \equiv \langle\sigma v\rangle_{\tilde{Z}_1}(T = m_{\tilde{Z}_1}/5 \times 10^{-5})$.

The thermal production rate for axions, saxions, axinos and gravitinos was computed in Refs.[5, 6, 7]. However, the expressions derived in Refs.[5, 6, 7] are only valid for out of equilibrium production, where the (out of equilibrium) thermal production rate (W_i) is defined by:

$$\frac{dn_i}{dt} + 3H = W_i \quad (3.2)$$

In the out of equilibrium regime we have $\bar{n}_i \gg n_i$, so Eq.(1.33) becomes:

$$\frac{dn_i}{dt} + 3H = \bar{n}_i^2 \langle\sigma v\rangle_i \quad (3.3)$$

Comparing the above equation to Eq.(3.2), we identify:

$$\langle\sigma v\rangle_i = \frac{W_i}{\bar{n}_i^2} \quad (3.4)$$

Although the above relation is only exact for the out of equilibrium regime, we use it for all values of temperature. This is only a poor approximation if the reheat temperature (T_R) is very close to the decoupling temperature (T_{dec}). Since for $T_R \gtrsim T_{dec}$, Eq.(1.33) gives $n_i = \bar{n}_i$ (independent of $\langle\sigma v\rangle_i$), while for $T_R \lesssim T_{dec}$ Eq.(3.4) is exact.

Using the expressions [5, 6, 7] and Eq.(3.4), we obtain²:

$$\begin{aligned} \langle\sigma v\rangle_a &= \frac{9}{128\pi^3\xi(3)} \frac{g_s^6}{f_a^2} \ln\left(\frac{1.0126}{g_s}\right) \\ \langle\sigma v\rangle_s &= \langle\sigma v\rangle_a \end{aligned}$$

²According to Ref.[5], the axion and saxion thermal rates are identical in supersymmetric models.

$$\begin{aligned}
\langle \sigma v \rangle_{\tilde{a}} &= \frac{1}{576\pi^3 \xi(3)^2} \frac{g_s^4}{f_a^2} F(g_s) \\
\langle \sigma v \rangle_{\tilde{G}} &= \frac{1.37}{M_P^2} \times \left[72g_s^2 \ln(1.271/g_s) \left(1 + \frac{M_3^2}{3m_{\tilde{G}}^2}\right) \right. \\
&\quad \left. + 27g^2 \ln(1.312/g) \left(1 + \frac{M_2^2}{3m_{\tilde{G}}^2}\right) + 11g'^2 \ln(1.266/g') \left(1 + \frac{M_1^2}{3m_{\tilde{G}}^2}\right) \right]
\end{aligned}$$

where $F(g_s)$ is numerically obtained from Ref.[6]:

$$F(x) = -0.365771 + 9.38897x + 27.7315x^2 - 20.1012x^3 + 5.153x^4 \quad (3.5)$$

We note that while the expression for $\langle \sigma v \rangle_{\tilde{a}}$ includes thermal decays such as $g \rightarrow \tilde{g} + \tilde{a}$ and is valid for all values of g_s , the expressions for $\langle \sigma v \rangle_{a,s}$ correspond only to the hard thermal loop calculation and assume $g_s \ll 1$ (or $T_R \gg 10^6$ GeV). Nonetheless we use these expressions for all values of g_s (T_R).

3.2 Γ, BR

The decay rates are computed through the SUBROUTINES:

AXINOB(R,MAXINO,FA,AXLT,AXWD,AXVIS)
 GRAVITINOB(R,MGT,GLT,GWD)
 Z1BRS(MZ1,FA,Z1B,Z1WD,Z1LT)
 SAXIONBR(MSAXION,FA,XI,SAXWD,SAXLT)

where the axino, gravitino, neutralino and saxion decay rates are given by AXWD, GWD, Z1WD and SAXWD, respectively. Neutralinos and axinos decays into gravitinos (if kinematically allowed) are NOT included. So the gravitino LSP case is NOT presently included. For the case of an axino LSP, Z1BRS assumes MAXINO=0. Therefore, if $m_{\tilde{Z}_1} < m_{\tilde{a}}$, the user should set Z1WD=0.

The branching ratios are given by:

$$\begin{aligned}
BR(s \rightarrow X) &= 1 - BR(s \rightarrow aa) - BR(s \rightarrow \tilde{a}\tilde{a}) \\
BR(s \rightarrow \tilde{Z}_1) &= 2 \times BR(s \rightarrow \tilde{g}\tilde{g}) \\
BR(s \rightarrow a) &= 2 \times BR(s \rightarrow aa)
\end{aligned}$$

- If $m_{\tilde{Z}_1} > m_{\tilde{a}}$:

$$\begin{aligned}
BR(\tilde{Z}_1 \rightarrow \tilde{a}) &= 1 \\
BR(\tilde{Z}_1 \rightarrow X) &= 1
\end{aligned}$$

- If $m_{\tilde{a}} > m_{\tilde{Z}_1}$:

$$\begin{aligned}
BR(\tilde{a} \rightarrow \tilde{Z}_1) &= 1 \\
BR(\tilde{a} \rightarrow X) &= 1
\end{aligned}$$

- If $m_{\tilde{G}} > m_{\tilde{Z}_1}$:

$$\begin{aligned} BR(\tilde{G} \rightarrow \tilde{Z}_1) &= 1 \\ BR(\tilde{G} \rightarrow X) &= 1 \end{aligned}$$

- If $m_{\tilde{G}} < m_{\tilde{Z}_1}$ (but $m_{\tilde{G}} > m_{\tilde{a}}$, since \tilde{G} LSP is not allowed!):

$$\begin{aligned} BR(\tilde{G} \rightarrow \tilde{a}) &= 1 \\ BR(\tilde{G} \rightarrow a) &= 1 \end{aligned}$$

and all other BRs are zero.

As discussed above, the branching ratio $BR(i \rightarrow X)$ is defined as the fraction of energy injected in the thermal bath. For simplicity we assume that most of the energy from decays goes into radiation, so $BR(i \rightarrow X) \sim 1$, except for saxion decays to axions and axinos and gravitino decays to axion + axino.

3.3 C

The source term C_i can be used to include other processes that can not be described as decays or annihilations. However these are not relevant for the PQMSSM. Therefore we set all these to zero.

3.4 ρ^0

Finally, the initial energy densities for the oscillating saxion and axion fields are given by:

$$\begin{aligned} \rho_a^0 &= 1.44 \frac{m_a(T)^2 f_a^2 \theta_i^2}{2} f(\theta_i)^{7/6} \\ \rho_s^0 &= \min \left[1.9 \times 10^{-8} \left(\frac{2\pi^2 g_*(T_R) T_R^3}{45} \right) \left(\frac{T_R}{10^5} \right) \left(\frac{s_i}{10^{12}} \right)^2, \frac{m_s^2 s_i^2}{2} \right] \end{aligned}$$

where $f(\theta_i) = \ln[e/(1 - \theta_i^2/\pi^2)]$ and $f_a \theta_i$ and s_i are the initial axion and saxion amplitudes. The definition of ρ_s^0 accounts for the possibility of beginning of saxion oscillations during inflation ($T_R < T_{osc}$).

References

- [1] E. Kolb and M. Turner, *The Early Universe*, Addison-Wesley Pub. (1990).
- [2] M. Kawasaki, G. Steigman and H. Kang, *Cosmological evolution of generic early-decaying particles and their daughters*, Nucl.Phys. B402, 323-348 (1993).
- [3] M. Kawasaki, G. Steigman and H. Kang, *Cosmological evolution of an early-decaying particle*, Nucl.Phys. B403, 671-706 (1993).
- [4] P. Graff and F. Steffen, *Thermal axion production in the primordial quark-gluon plasma*, Phys.Rev. D 83, 070511 (2011).
- [5] P. Graff and F. Steffen, *Axions and saxions from the primordial supersymmetric plasma and extra radiation signatures*, arXiv:1208.2951 (2012).

- [6] A. Strumia, *Thermal production of axino Dark Matter*, JHEP 06, 036 (2010).
- [7] J. Pradler and F. Steffen, *Constraints on the reheating temperature in gravitino dark matter scenarios*, Phys.Lett. **B 648**, 224 (2007).