diffusions Documentation

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Stanislav Khrapov

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CHAPTER 1

General Affine Diffusions

A jump-diffusion process is a Markov process solving the stochastic differential equationd

$$Y_t = \mu\left(Y_t, \theta_0\right) dt + \sigma\left(Y_t, \theta_0\right) dW_t.$$

A discount-rate function $R : D \to \mathbb{R}$ is an affine function of the state

$$R\left(Y\right) = \rho_0 + \rho_1 \cdot Y,$$

for $\rho = (\rho_0, \rho_1) \in \mathbb{R} \times \mathbb{R}^N$.

The affine dependence of the drift and diffusion coefficients of Y are determined by coefficients (K, H) defined by:

$$\mu\left(Y\right) = K_0 + K_1 Y,$$

for $K = (K_0, K_1) \in \mathbb{R}^N \times \mathbb{R}^{N \times N}$,

and

$$\left[\sigma\left(Y\right)\sigma\left(Y\right)'\right]_{ij} = \left[H_0\right]_{ij} + \left[H_1\right]_{ij} \cdot Y,$$

for $H = (H_0, H_1) \in \mathbb{R}^{N \times N} \times \mathbb{R}^{N \times N \times N}$.

Here

$$\left[H_1\right]_{ij} \cdot Y = \sum_{k=1}^{N} \left[H_1\right]_{ijk} Y_k.$$

A characteristic $\chi = (K, H, \rho)$ captures both the distribution of Y as well as the effects of any discounting.

CHAPTER 2

Geometric Brownian Motion (GBM)

Suppose that S_t evolves according to

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

In logs:

$$d \log S_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t.$$

After integration on the interval $[t, t+h]$:

$$r_{t,h} = \log \frac{S_{t+h}}{S_t} = \left(\mu - \frac{1}{2}\sigma^2\right) h + \sigma \sqrt{h} \varepsilon_{t+h},$$

where $\varepsilon_t \sim N(0, 1)$.

CHAPTER 3

Vasicek

Suppose that r_t evolves according to

$$dr_t = \kappa\left(\mu - r_t\right)dt + \eta dW_t.$$

CHAPTER 4

Cox-Ingersoll-Ross (CIR)

Suppose that r_t evolves according to

$$dr_t = \kappa\left(\mu - r_t\right)dt + \eta\sqrt{r_t}dW_t.$$

Feller condition for positivity of the process is $\kappa\mu > \frac{1}{2}\eta^2$.

CHAPTER 5

Heston

The model is

$$dp_t = \left(r + \left(\lambda_r - \frac{1}{2}\sigma_t^2\right)\right) dt + \sigma_t dW_t^r,$$
$$d\sigma_t^2 = \kappa\left(\mu - \sigma_t^2\right) dt + \eta\sigma_t dW_t^\sigma,$$

with $p_t = \log S_t$, and $Corr\left[dW_s^r, dW_s^\sigma\right] = \rho$, or in other words

$$W_t^\sigma = \rho W_t^r + \sqrt{1-\rho^2} W_t^v.$$

Feller condition for positivity of the volatility process is $\kappa\mu > \frac{1}{2}\eta^2$.

CHAPTER 6

Central Tendency (CT)

The model is

$$
\begin{aligned}
dp_t &= \left(r + \left(\lambda - \frac{1}{2}\right)\sigma_t^2\right)dt + \sigma_t dW_t^r,\\
d\sigma_t^2 &= \kappa_\sigma\left(v_t^2 - \sigma_t^2\right)dt + \eta_\sigma \sigma_t dW_t^\sigma,\\
dv_t^2 &= \kappa_v\left(\mu - v_t^2\right)dt + \eta_v v_t dW_t^v,
\end{aligned}
$$

with $p_t = \log S_t$, and $Corr\left[dW_s^r, dW_s^\sigma\right] = \rho$, or in other words $W_t^\sigma = \rho W_t^r + \sqrt{1-\rho^2}W_t^v$. Also let $R\left(Y_t\right) = r$.