nbcalc

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Contents

1 Indices and tables

```
In [1]: %matplotlib inline
        import matplotlib.pyplot as plt
        import numpy as np
        import sympy as sy
```

1.1 Composition and the Chain Rule

As you may have guessed, once we apply the idea of differentiation to simple curves, we'd like to use this method in more and more general situations. This notebook explores the use of differentiation and derivatives to explore functions formed by composition, and to use these results to differentiate implicitly defined functions.

GOALS:

- Identify situations where chain rule is of use
- Define and Use Chain Rule
- Use Chain Rule to differentiate implicitly defined functions
- Use Descartes algorithm to explore alternative approach to differentation

Composition of Functions

Functions can be formed by combining other functions through famililar operations. For example, we can consider the polynomial $h(x) = x^3 + x^2$ as formed by two simpler polynomials $f(x) = x^3$ and $g(x) = x^2$ combined through addition. So far, we have not had to worry about this, as differentiation and integration are linear operators that work across addition and subtraction.

If we instead have a function h given by:

$$h(x) = \sqrt{x^3 + x^2}$$

we may recognize the square root function and the polynomial inside of it. This was not formed by addition, subtraction, multiplication, or division of simpler functions however. Instead, we can understand the function h as formed by **composing** two functions f and g where:

$$h(x) = \sqrt{x^3 + x^2} \quad f(u) = \sqrt{(u)} \quad g(x) = x^3 + x^2$$

The operation of composition means we apply the function f to the function g. We would write this as

$$f(g(x)) = \sqrt{(g(x))} = \sqrt{x^3 + x^2}$$

We can use SymPy to explore a few examples and determine a general rule for differentiating functions formed by compositions. We begin with trying to generalize the situation above, where we compose some function g into a function f of the form

$$f(x) = (g(x))^n$$

It seems reasonable to expect that

$$f'(x) = n(g(x))^{n-1}$$

You need to adjust this statement to make it true. Consider the following examples, use sympy to differentiate them and determine the remaining terms.

1. $(x^2 - 3x)^2$
2. $(x^2 - 3x)^3$
3. $\sqrt{x^2 - 3x}$

```
In [2]: x = sy.Symbol('x')
        y1 = (x**2 - 3*x)**2
        y2 = (x**2 - 3*x)**3
        y3 = (x**2 - 3*x)**(1/2)

In [3]: dy1 = sy.diff(y1, x)
        sy.factor(dy1)

Out[3]: 2*x*(x - 3)*(2*x - 3)

In [4]: dy2 = sy.diff(y2, x)
        sy.factor(dy2)

Out[4]: 3*x**2*(x - 3)**2*(2*x - 3)

In [5]: dy3 = sy.diff(y3, x)
        dy3

Out[5]: (1.0*x - 1.5)*(x**2 - 3*x)**(-0.5)
```

Now consider the examples for $f(x) = \sin(g(x))$:

1. $\sin 2x$
2. $\sin\left(\frac{1}{2}x + 3\right)$
3. $\sin\left(x^2\right)$

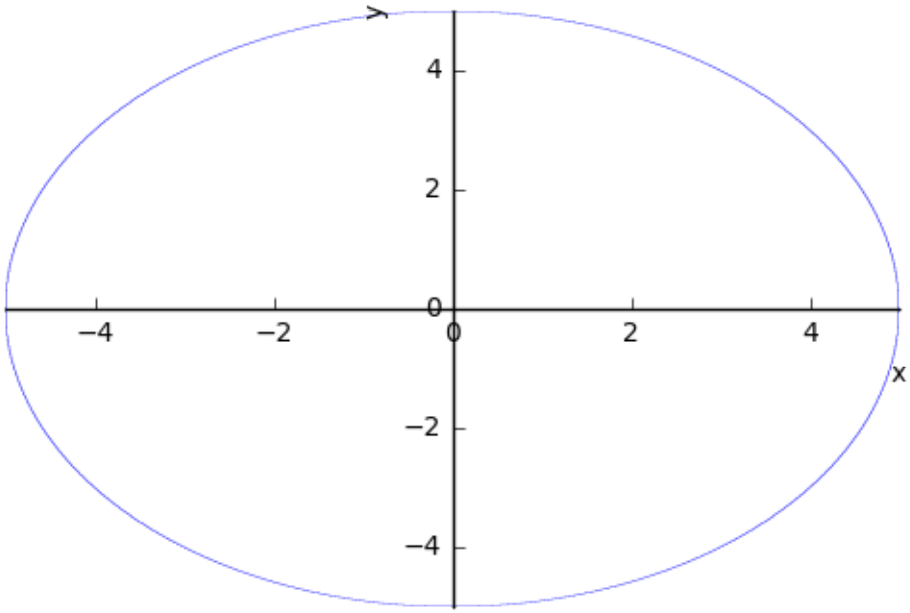
1.2 Implicitly Defined Functions

Rather than being given a relationship that can be expressed in terms of a single variable, we can also apply the technique of differentiation to implicitly defined curves. For example, consider the equation for a circle centered at the origin with radius 5, i.e.:

$$x^2 + y^2 = 25$$

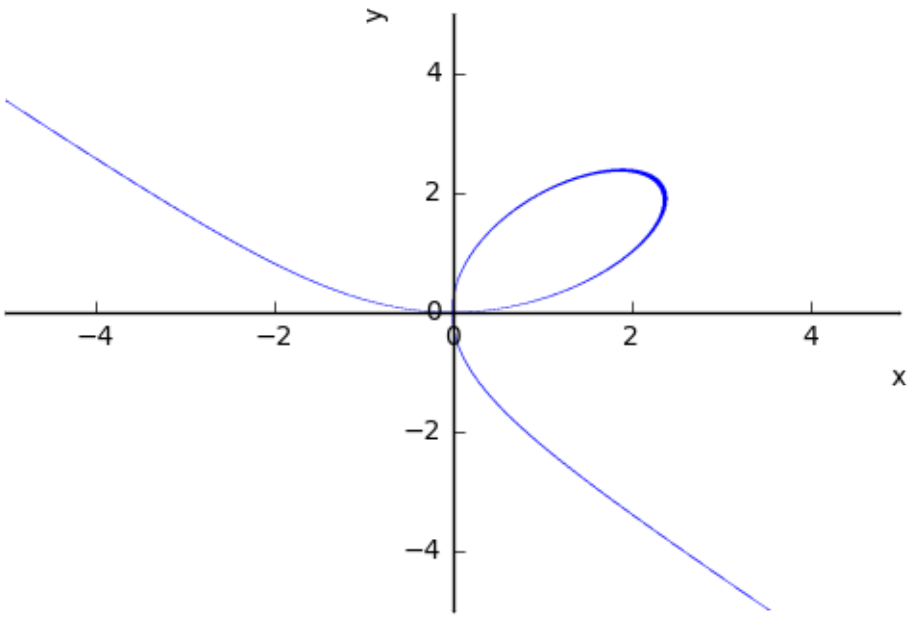
We are unable to express y in terms of x with a single equation. We should still be able to understand things like tangent lines however, and apply the idea of differentiation to the expression. We can plot implicitly defined functions with Sympy as shown below. These should look familiar to our parametric plots, and we could introduce parameters into these equations to describe the curves parametrically.

```
In [6]: x, y = sy.symbols('x y')

In [7]: y1 = sy.Eq(x**2 + y**2, 25)
        sy.plot_implicit(y1)
```



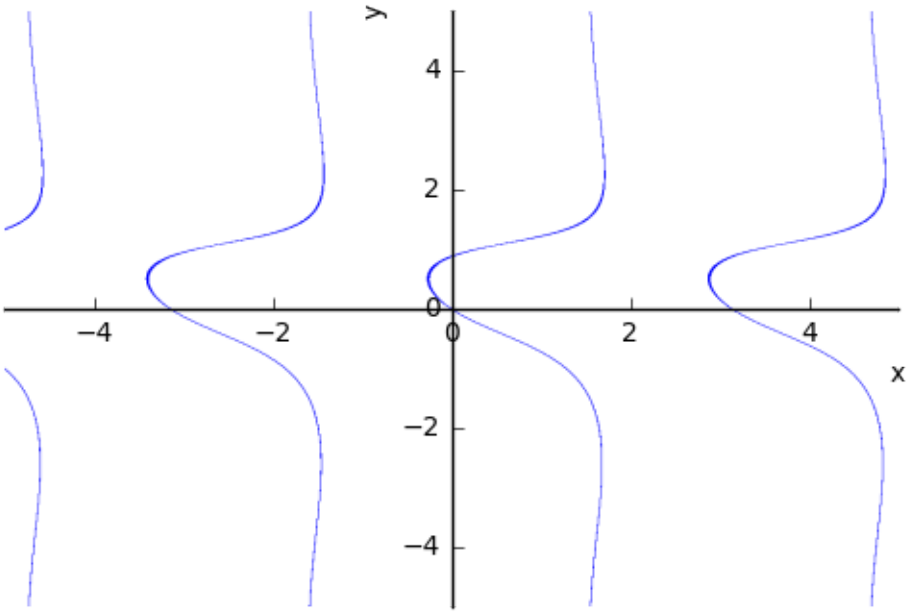
Out[7]: <sympy.plotting.plot.Plot at 0x1144b0780>

In [8]:
```
y2 = sy.Eq(x**3 + y**3, (9/2)*x*y)
sy.plot_implicit(y2)
```



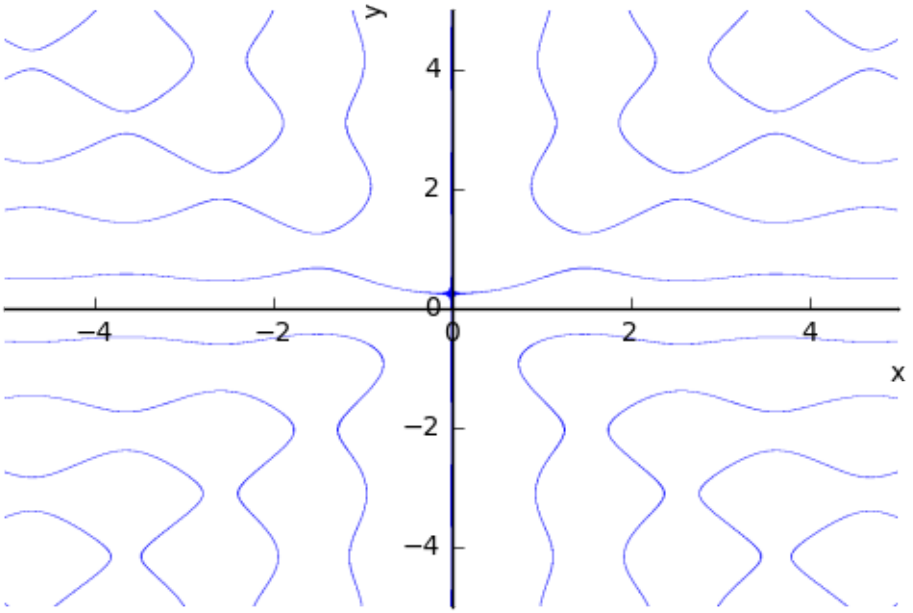
Out[8]: <sympy.plotting.plot.Plot at 0x1168e74a8>

In [9]:
```
y3 = sy.Eq(sy.sin(x + y), y**2*sy.cos(x))
sy.plot_implicit(y3)
```



Out[9]: <sympy.plotting.plot.Plot at 0x1169b3908>

In [10]:
```
y4 = sy.Eq(y*sy.sin(3*x), x*sy.cos(3*y))
sy.plot_implicit(y4)
```



Out[10]: <sympy.plotting.plot.Plot at 0x116a4cef0>

Algorithmic Implicit Differentiation

Hopefully, you recognize the problem. There are many tangents at a given value of x. However, we can determine a point on the graph and recall what we know about equations of lines. Take the second example, called the **Folium of Descartes**, defined as:

$$x^3 + y^3 = \frac{9}{2}xy$$

We know the point $(2,1)$ is on the graph. Also, we know that if we have a tangent line at this point, it would be a linear equation with some slope m that passes through $(2,1)$. This means

$$y = m(x-2)+1$$

These equations agree at $(2,1)$ so we can substitute the second into the first for y

$$x^3 + (m(x-2)+1)^3 = \frac{9}{2}x(m(x-2)+1)$$

There is some algebra involved here that we will call on Sympy to help us with. Here's the idea. From the picture, we see there will be two tangent lines at $x = 2$. This means our equation has a double root there, or that $(x-2)^2$ is a factor of this. We can eliminate one of these roots by dividing the expression by $x-2$, then substituting $x = 2$ into the remaining expression and solve for the slope m.

```
In [24]: x, y, m = sy.symbols('x y m')
         expr = x**3 + y**3 -(9/2)*x*y
         tan = m*(x-2) + 1

In [25]: expr = expr.subs(y, tan)
         sy.pprint(expr)

 3                                              3
x  - 4.5x(m(x - 2) + 1) + (m(x - 2) + 1)

In [26]: expr

Out[26]: x**3 - 4.5*x*(m*(x - 2) + 1) + (m*(x - 2) + 1)**3

In [31]: now = sy.quo(expr, x-2)
         now = now.subs(x, 2)

In [32]: sy.solve(now, m)

Out[32]: [1.25000000000000]
```

Textbook Approach

If we want to work directly on the expression, we can recall that y can be considered as a function of x, $y = f(x)$. When we do express things this way, the equation for the Folium above becomes

$$x^3 + (f(x))^3 = 6x(f(x))$$

The left hand side of the equation has the term x^3, whose derivative with respect to x is $3x^2$ from our familiar rules, and then our second term, $(f(x))^3$ makes use of the chain rule and we would get:

$$\frac{d}{dx}(f(x))^3 = 3\,(f(x))^2\,f'(x)$$

We treat the other elements of the expression similarly, and solve for the term f'. We can use Sympy to simplify these computations and verify the result above. The `idiff` function from sympy takes the implicit derivative with respect to the second variable. We can then evaluate the generale derivative at our point $(2,1)$.

```
In [85]: x, y = sy.symbols('x y')
         foli = x**3 + y**3 - (9/2)*x*y
         dx_foli = sy.idiff(foli, y, x)

In [86]: dx_foli.subs(x, 2).subs(y, 1)

Out[86]: 1.25000000000000
```

Problems

```
In [1]: %matplotlib inline
        import matplotlib.pyplot as plt
        import numpy as np
        import sympy as sy
        from scipy import interpolate, stats
```

1.3 Regression and Least Squares

Earlier, we encountered interpolation as a means to determine a line given only a few points. Interpolation determined a polynomial that would go through each point. This is not always sensible however.

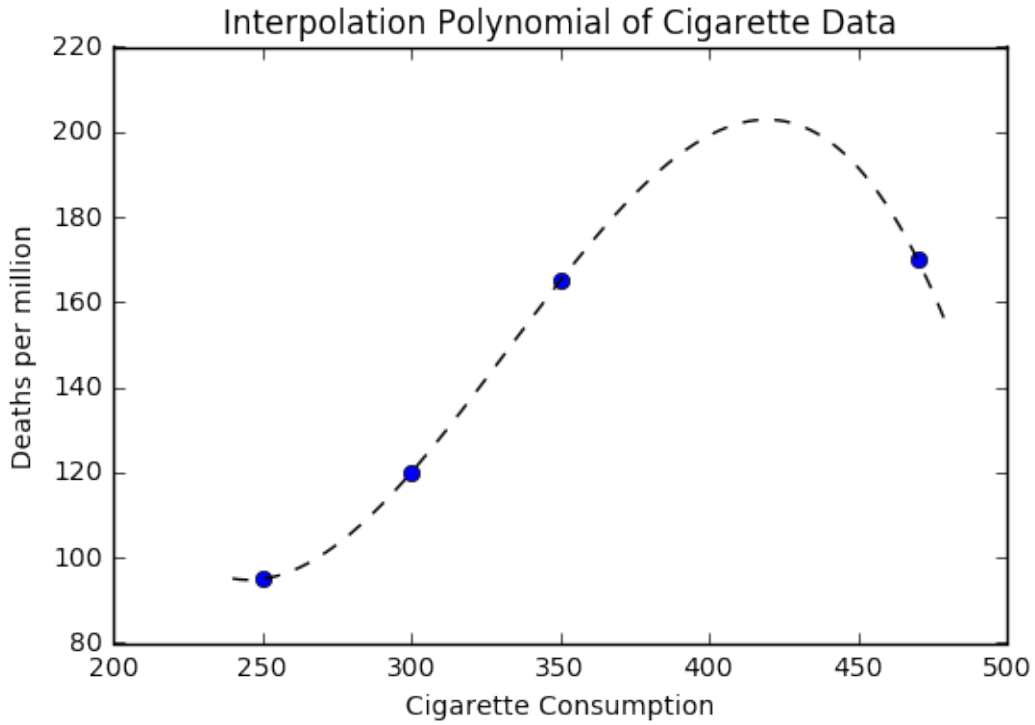
Let's consider the following example where the data represents cigarette consumption and death rates for countries given.

Country	Cigarette Consumption	Deaths per Million
Norway	250	95
Sweden	300	120
Denmark	350	165
Australia	470	170

If we use interpolation as we had earlier, we get the following picture.

```
In [2]: cigs = [250, 300, 350, 470]
        death = [95, 120, 165, 170]

        xs = np.linspace(240, 480, 1000)
        i1 = interpolate.splrep(cigs, death, s=0)
        ynews = interpolate.splev(xs, i1, der = 0)


        plt.plot(cigs, death, 'o')
        plt.plot(xs, ynews, '--k')
        plt.title("Interpolation Polynomial of Cigarette Data")
        plt.xlabel("Cigarette Consumption")
        plt.ylabel("Deaths per million")
```

```
Out[2]: <matplotlib.text.Text at 0x10fdfd400>
```



Now suppose that we wanted to use our polynomial to make a prediction about a country with higher cigarette consumption than that of Australia. You should notice that our polynomial would provide an estimate that is lower.

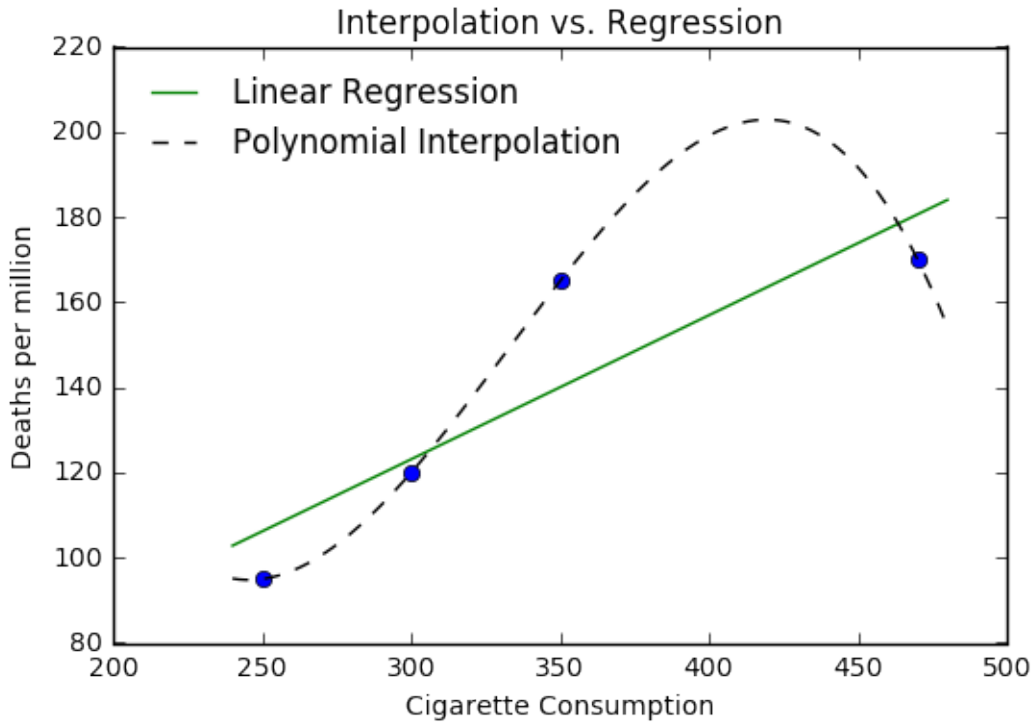
Alternatively, we can fit a straight line to the data with a simple numpy function `polyfit` where we describe the data we are fitting and the degree polynomial we are fitting. Here we want a straight line so we choose degree 1.

```
In [3]: a, b = np.polyfit(cigs, death, 1)
```

```
In [4]: def l(x):
            return a*x + b
```

```
In [5]: xs = np.linspace(240, 480, 1000)
        i1 = interpolate.splrep(cigs, death, s=0)
        ynews = interpolate.splev(xs, i1, der = 0)
        plt.plot(cigs, death, 'o')
        plt.plot(xs, l(xs), label = 'Linear Regression')
        plt.plot(xs, ynews, '--k', label = 'Polynomial Interpolation')
        plt.title("Interpolation vs. Regression")
        plt.xlabel("Cigarette Consumption")
        plt.ylabel("Deaths per million")
        plt.legend(loc = 'best', frameon = False)
```

```
Out[5]: <matplotlib.legend.Legend at 0x110456d30>
```

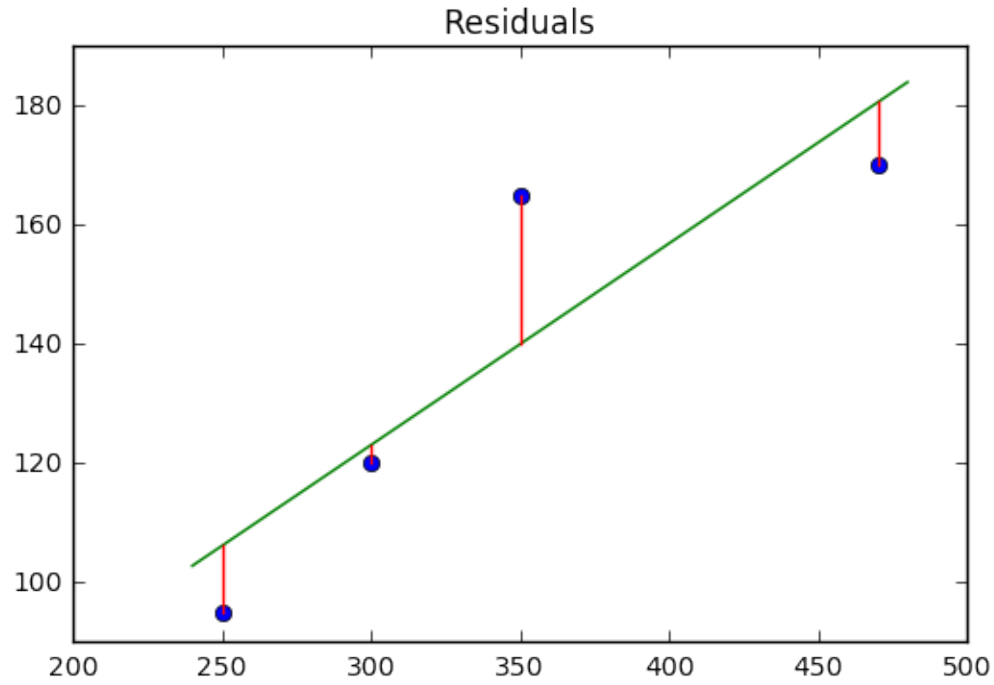


Determining the Line of Best Fit

The idea behind our line of best fit, is that it minimizes the distance between itself and all the data points. These distances are called residuals and are shown in the plot below.

```
In [6]: plt.plot(cigs, death, 'o')
        plt.plot(xs, l(xs), label = 'Linear Regression')
        plt.plot((cigs[0], cigs[0]), (death[0], l(cigs[0])), c = 'red')
        plt.plot((cigs[1], cigs[1]), (death[1], l(cigs[1])), c = 'red')
        plt.plot((cigs[2], cigs[2]), (death[2], l(cigs[2])), c = 'red')
        plt.plot((cigs[3], cigs[3]), (death[3], l(cigs[3])), c = 'red')
        plt.title("Residuals")
```

```
Out[6]: <matplotlib.text.Text at 0x1104bc710>
```



The line of best fit minimizes these distances using familiar techniques of differentiation that we have studied. First, we investigate the criteria of least squares, that says the residuals are minimized by finding the smallest **ROOT MEAN SQUARE ERROR** or **RMSE**.

In general, we see that a residual is the distance between some actual data point (x_i, y_i) and the resulting point on the line of best fit $l(x)$ at the point $(x_i, l(x_i))$.

Suppose we were deciding between the lines

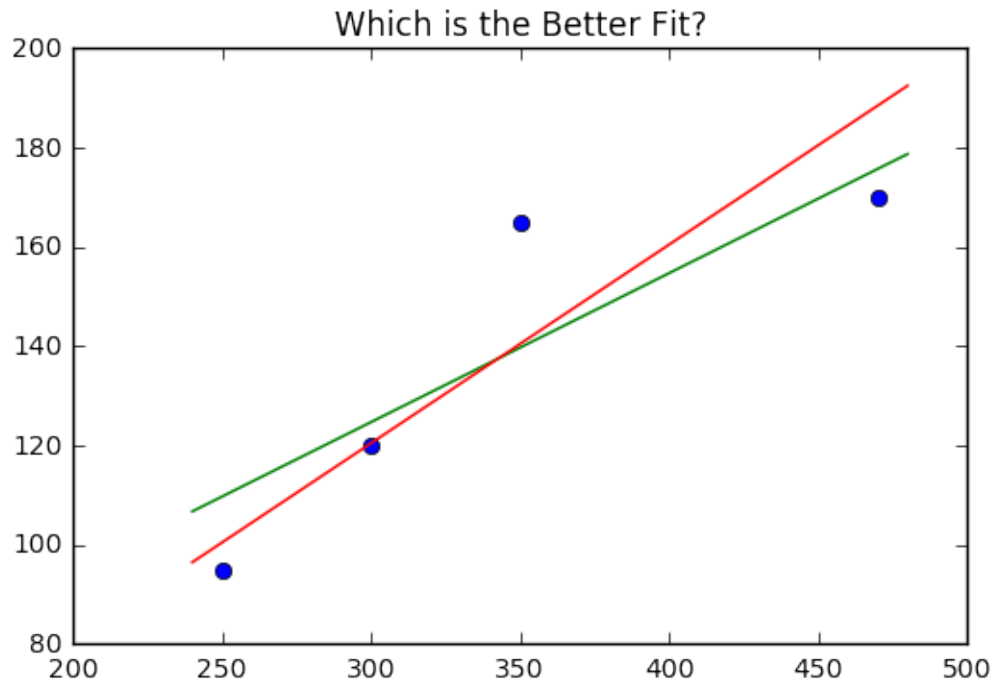
$$y_1 = .3x + 34.75 \quad y_2 = .4x + .5$$

We want to compare the average difference between the actual and predicted values. We can find the residuals by creating a list of differences in terms of actual and predicted values.

```
In [7]: def y1(x):
            return 0.3*x + 34.75

        def y2(x):
            return 0.4*x + 0.5
```

```
In [8]: plt.plot(cigs, death, 'o')
        plt.plot(xs, y1(xs))
        plt.plot(xs, y2(xs))
        plt.title("Which is the Better Fit?")
```

```
Out[8]: <matplotlib.text.Text at 0x110609e48>
```




```
In [9]: resid_1 = [np.sqrt((death[i] - y1(cigs[i]))**2) for i in range(len(cigs))]
        resid_2 = [np.sqrt((death[i] - y2(cigs[i]))**2) for i in range(len(cigs))]

In [10]: resid_1

Out[10]: [14.75, 4.75, 25.25, 5.75]

In [11]: resid_2

Out[11]: [5.5, 0.5, 24.5, 18.5]

In [12]: resid_1_sq = [(a**2 )/4for a in resid_1]
         resid_2_sq = [(a**2)/4 for a in resid_2]

In [13]: np.sqrt(sum(resid_1_sq))

Out[13]: 15.089317413322579

In [14]: np.sqrt(sum(resid_2_sq))

Out[14]: 15.596473960482221
```

Thus, the first line $y1$ is considered a better fit for the data.

Deriving the Equation of the Line

In general, we have some line of best fit y given by:

$$y = a + bx$$

If we have some set of points $(x_1, y_1), (x_2, y_2), (x_3, y_3)...(x_n, y_n)$. We need to minimize the sum of squares of residuals here, so we would have a number of values determined by:

$$[y_1 - (a + bx_1)]^2 + [y_2 - (a + bx_2)]^2 + [y_3 - (a + bx_3)]^2 + ...$$

which we can rewrite in summation notation as

$$\sum_{i=1}^{n} [y_i - (a + bx_i)]^2$$

We can consider this as a function in terms of the variable a that we are seeking to minimize.

$$g(a) = \sum_{i=1}^{n} [y_i - (a + bx_i)]^2$$

From here, we can apply our familiar strategy of differentiating the function and locating the critical values. We are looking for the derivative of a sum, which turns out to be equivalent to the sum of the derivatives, hence we have

$$g'(a) = \sum_{i=1}^{n} \frac{d}{da}[y_i - (a + bx_i)]^2$$

$$g'(a) = \sum_{i=1}^{n} 2[y_i - a - bx_i](-1)$$

$$g'(a) = -2[\sum_{i=1}^{n} y_i - a - b\sum_{i=1}^{n} x_i]$$

Setting this equal to zero and solving for a we get

$$a = \frac{1}{n}\sum_{i=1}^{n} y_i - b\frac{1}{n}\sum_{i=1}^{n} x_i$$

The terms should be familiar as averages, and we can rewrite our equation as

$$a = \bar{y} - b\bar{x}$$

We now use this to investigate a similar function in terms of b to complete our solution.

$$f(b) = \sum_{i=1}^{n} [y_i - (\bar{y} + b(x_i - \bar{x}))]^2$$

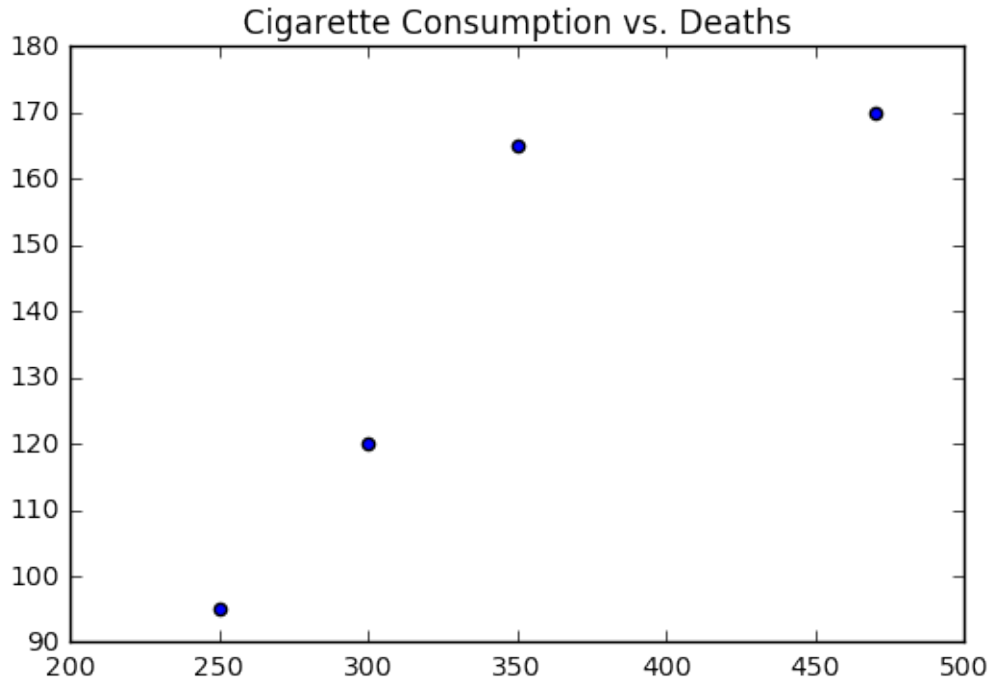
We end up with

$$b = \sum_{i=1}^{n} \frac{(x_i - \bar{x})(y_i - \bar{y})}{(\bar{x} - x_i)^2}$$

Let's return to the problem of cigarette consumption and test our work out by manually computing a and b.

```
In [15]: cigs = [250, 300, 350, 470]
         death = [95, 120, 165, 170]

         plt.scatter(cigs, death)
         plt.title("Cigarette Consumption vs. Deaths")
```

```
Out[15]: <matplotlib.text.Text at 0x110677710>
```

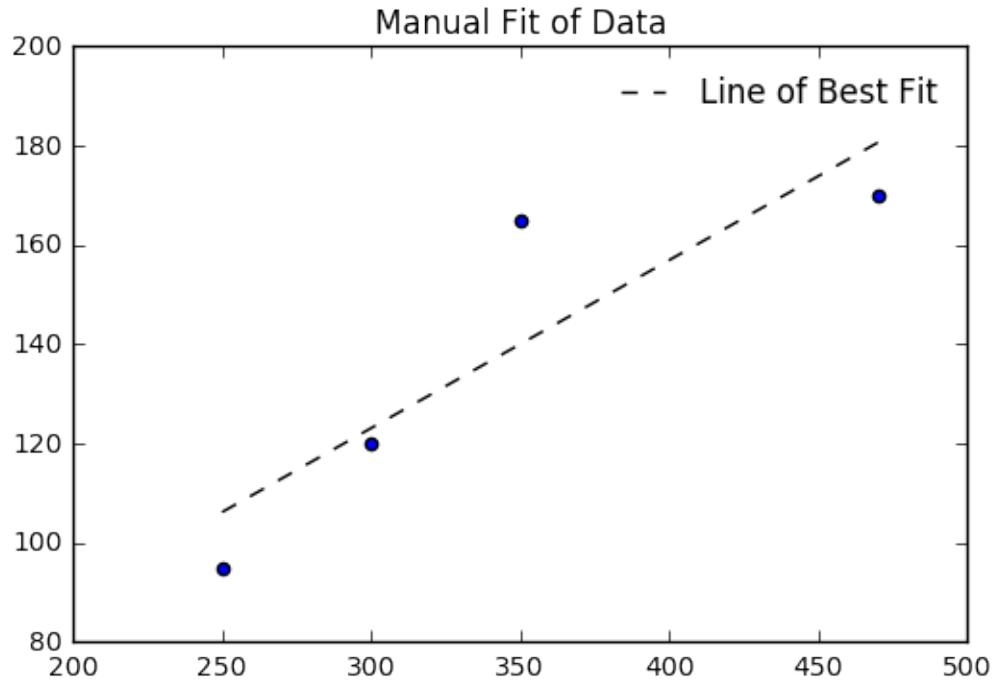


In [16]:
```
ybar = np.mean(death)
xbar = np.mean(cigs)
ydiff = (death - ybar)
xdiff = (cigs - xbar)

b = np.sum(ydiff*xdiff)/np.sum(xdiff**2)
a = ybar - b*xbar
a, b
```

Out[16]: (21.621368322399249, 0.33833177132146203)

In [17]:
```
l = [a + b*i for i in cigs]
plt.scatter(cigs, death)
plt.plot(cigs, l, '--k', label = 'Line of Best Fit')
plt.title("Manual Fit of Data")
plt.legend(loc = 'best', frameon = False)
```

Out[17]: <matplotlib.legend.Legend at 0x110898860>



We can check with numpy and see if our values for a and b agree with the computer model.

```
In [18]: b2, a2 = np.polyfit(cigs, death, 1)
         a2, b2
```

```
Out[18]: (21.621368322399242, 0.33833177132146203)
```

Finally, we can write a simple function to compute the RMSE.

```
In [19]: l
```

```
Out[19]: [106.20431115276476, 123.12089971883786, 140.03748828491098, 180.6373008434864]
```

```
In [20]: death
```

```
Out[20]: [95, 120, 165, 170]
```

```
In [21]: diff = [(l[i] - death[i])**2 for i in range(len(l))]
         diff
```

```
Out[21]: [125.53658840796878,
          9.7400150550422442,
          623.12699112595669,
          113.15216923483649]
```

```
In [22]: np.sqrt(np.sum(diff)/len(l))
```

```
Out[22]: 14.761061647318972
```

Other Situations

Our goal with regression is to identify situations where regression makes sense, fit models and discuss the reasonableness of the model for describing the data. Data does not always come in linear forms however.

We can easily generate sample data for familiar curves. First, we can make some lists of polynomial form, then we will add some noise to these, fit models with `np.polyfit()`, and plot the results.

Non-Linear Functions

Plotting and fitting non-linear functions follows a similar pattern, however we need to take into consideration the nature of the function. First, if we see something following a polynomial pattern, we can just use whatever degree polynomial fit we believe is relevant. The derivation of these formulas follows the same structure as the linear case, except you are replacing the line $a - bx_i$ with a polynomial $a + bx_i + cx_i^2$....

If we believe there to be an exponential fit, we can transform this into a linear situation using the logarithm. For example, suppose we have the following population data.

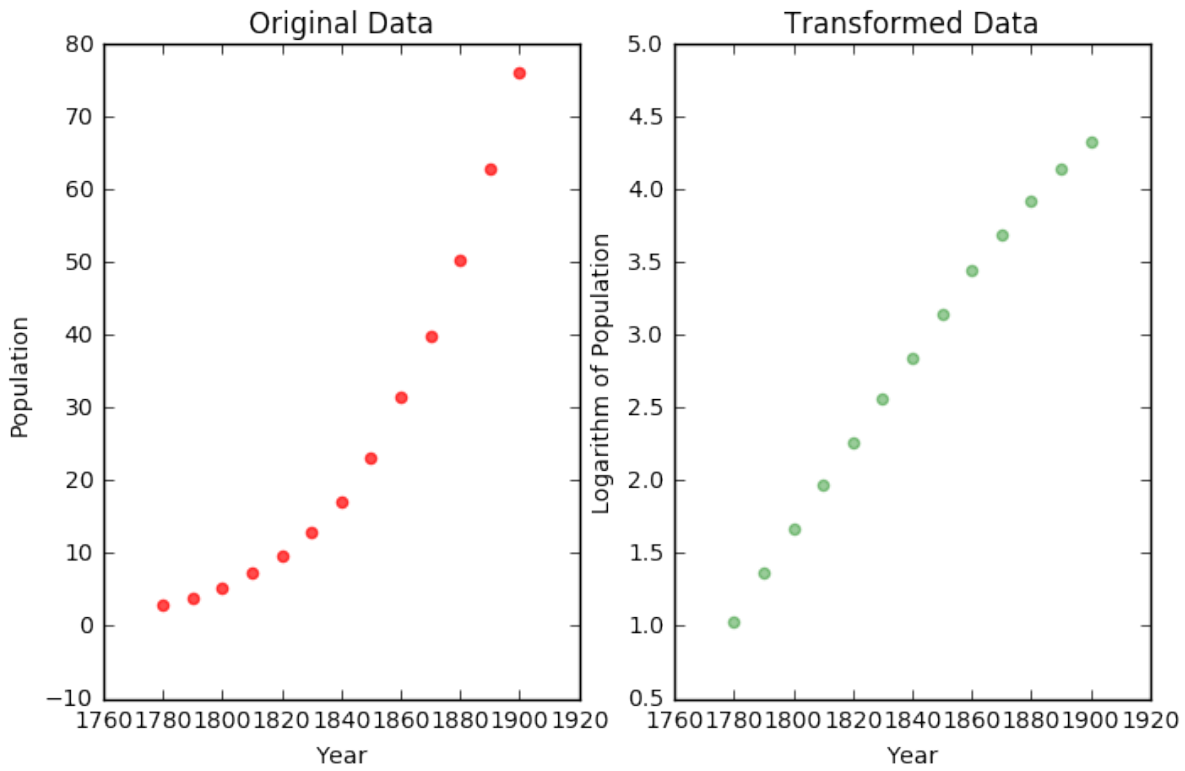
Decade t	Year	Population
0	1780	2.8
1	1790	3.9
2	1800	5.3
3	1810	7.2

If we examine the data, we see an exponential like trend. If we use NumPy to find the logarithm of the population values and plot the result, we note the transformed datas similarity to a linear function.

```
In [23]: t = np.arange(0,13)
         year = np.arange(1780,1910,10)
         P = [2.8, 3.9, 5.3, 7.2, 9.6, 12.9, 17.1, 23.2, 31.4, 39.8, 50.2, 62.9, 76.0]
```

```
In [24]: plt.figure(figsize = (8,5))
         plt.subplot(1, 2, 1)
         plt.scatter(year, P,color = 'red', alpha = 0.7)
         plt.xlabel('Year')
         plt.ylabel('Population')
         plt.title("Original Data")

         plt.subplot(1, 2, 2)
         lnP = np.log(P)
         plt.scatter(year, lnP, color = 'green', alpha = 0.4)
         plt.xlabel('Year')
         plt.ylabel('Logarithm of Population')
         plt.title("Transformed Data")
```

```
Out[24]: <matplotlib.text.Text at 0x110a026d8>
```



Symbolically, we would imagine the original function as an exponential of the form

$$y = ae^{bx}$$

The expression can be explored in a similar manner, where we use Sympy to find the effect of the logarithm.

```
In [25]: y, a, b, x = sy.symbols('y a b x')

In [26]: eq = sy.Eq(y, a*sy.exp(b*x))

In [27]: sy.expand_log(sy.log(b**x))

Out[27]: log(b**x)

In [28]: sy.expand_log(sy.log(a*sy.exp(b*x)), force = True)

Out[28]: b*x + log(a)
```

Hence, we have that

$$\log(y) = bx + \log(a)$$

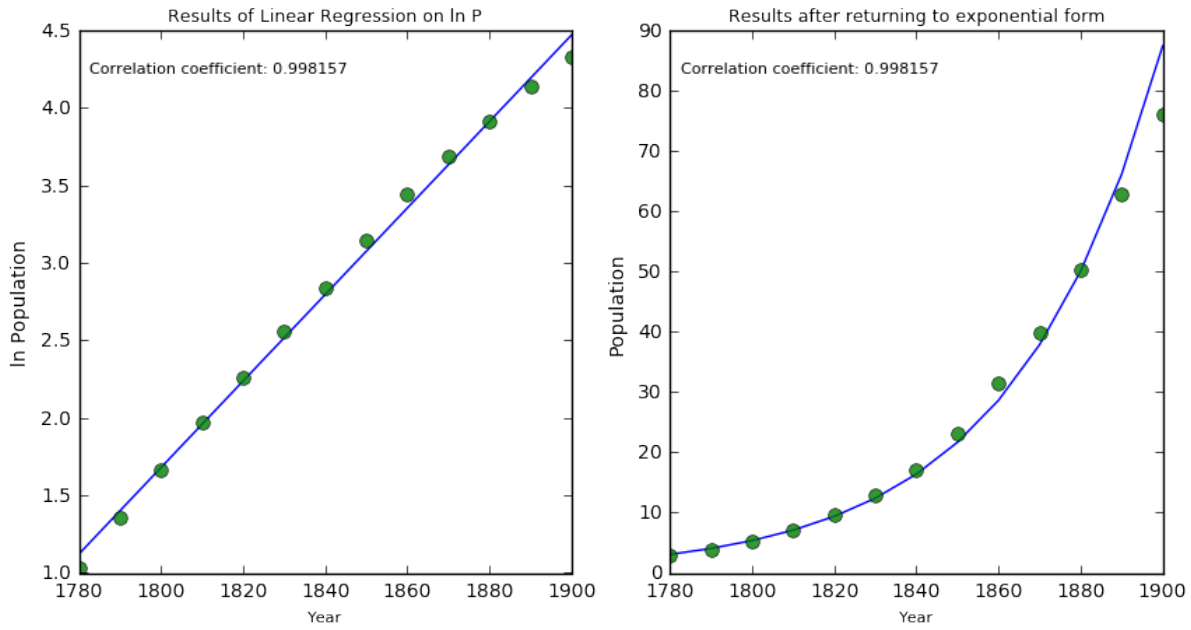
which should look like our familiar linear equations. Here, we can find a and b, then convert the equation back to its original form by undoing the logarithm with the exponential.

For kicks, we introduce the SciPy `linregress` function. Feel free to examine the help documentation for the function. This gives a little more information about the model than the `polyfit` function. Further, we add text to the plot to display information about the model.

```
In [29]: line = np.polyfit(year, lnP, 1)
         fit = np.polyval(line, year)
         alpha, beta, r_value, p_value, std_err = stats.linregress(year, lnP) #
         alpha, beta, r_value

Out[29]: (0.027906119028040688, -48.55083021891685, 0.99815691149842678)

In [30]: fig = plt.figure(figsize = (10,5))
```

```
plt.subplot(1, 2, 2)
plt.plot(year, np.exp(fit))
plt.plot(year, P, 'o', markersize = 7, alpha = 0.8)
ax = fig.add_subplot(121)
text_string = "\nCorrelation coefficient: %f" % (r_value)
ax.text(0.022, 0.972, text_string, transform=ax.transAxes, verticalalignment='top',  fontsize=8)
plt.title("Results of Linear Regression on ln P", fontsize = 9)
plt.xlabel("Year", fontsize = 8)
plt.ylabel("ln Population")

plt.subplot(1, 2, 1)
plt.plot(year, fit)
plt.plot(year, lnP, 'o', markersize = 7, alpha = 0.8)
ax = fig.add_subplot(122)
text_string = "\nCorrelation coefficient: %f" % (r_value)
ax.text(0.022, 0.972, text_string, transform=ax.transAxes, verticalalignment='top',  fontsize=8)
plt.title("Results after returning to exponential form", fontsize = 9)
plt.xlabel("Year", fontsize = 8)
plt.ylabel("Population")
```

Out[30]: <matplotlib.text.Text at 0x110c1c748>



Logistic Example

As you see above, towards the end of our model the actual and predicted values seem to diverge. Considering the context, this makes sense. A population should reach some maximum levels due to physical resources. A more S shaped curve is the logistic function which is given by

$$y = \frac{L}{1 + e^{a+bx}}$$

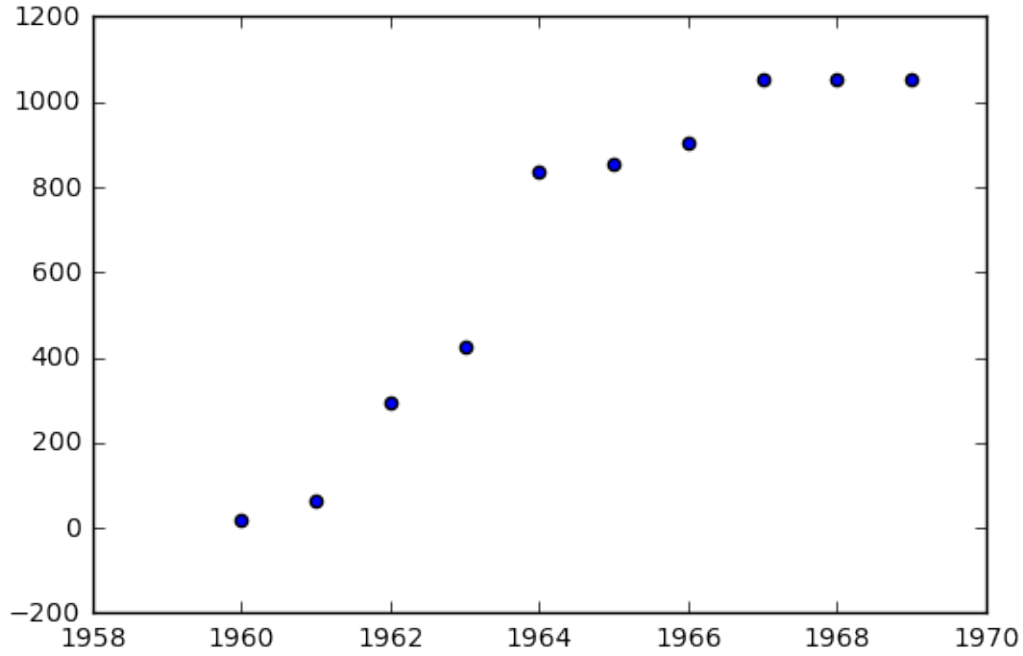
As an example, consider the Inter Continental Ballistic Missle Data for 1960 - 1969.

Year	Number of ICBM's
1960	18
1961	63
1962	294
1963	424
1964	834
1965	854
1966	904
1967	1054
1968	1054
1969	1054

```
In [31]: year = [i for i in np.arange(1960, 1970, 1)]
         icbm = [18, 63, 294, 424, 834, 854, 904, 1054, 1054, 1054]
```

```
In [32]: plt.scatter(year, icbm)
```

```
Out[32]: <matplotlib.collections.PathCollection at 0x1107ab748>
```




```
In [33]: L, y, a, b, x = sy.symbols('L y a b x')
```

```
In [34]: exp = sy.Eq(y, L/(1 + sy.exp(a + b*x)))
```

```
In [35]: sy.solve(exp, (a + b*x),  force = True)
```

```
Out[35]: [log((L - y)/y)]
```

This means that the tranformation that linearizes our data is

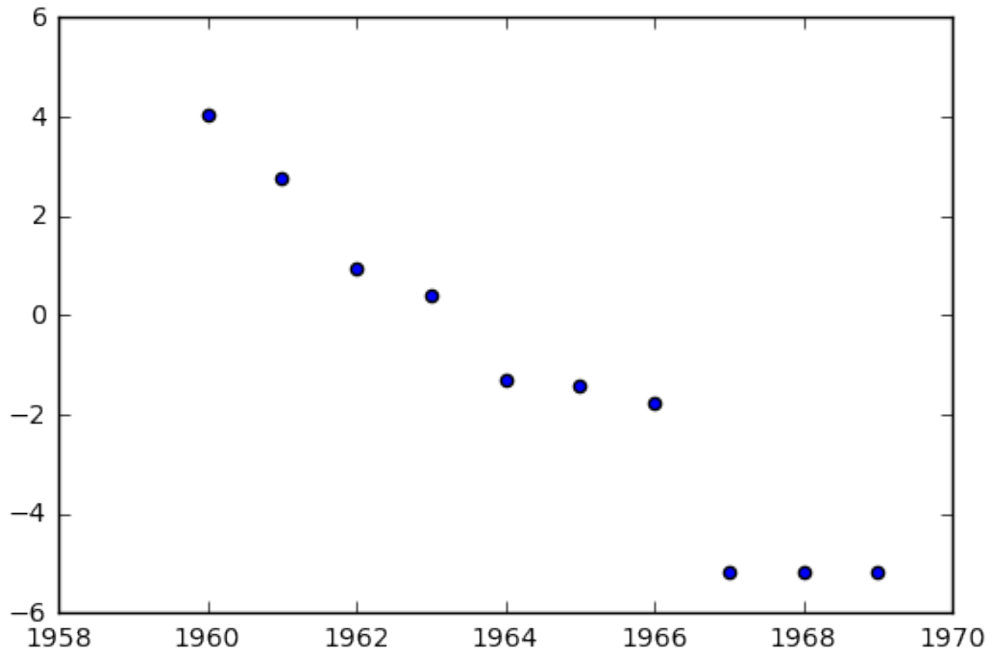
$$\log(\frac{L-y}{y})$$

The value L is defined as the *carrying capacity* of the model. Here, it seems something like $L = 1060$ would be a reasonable value to try.

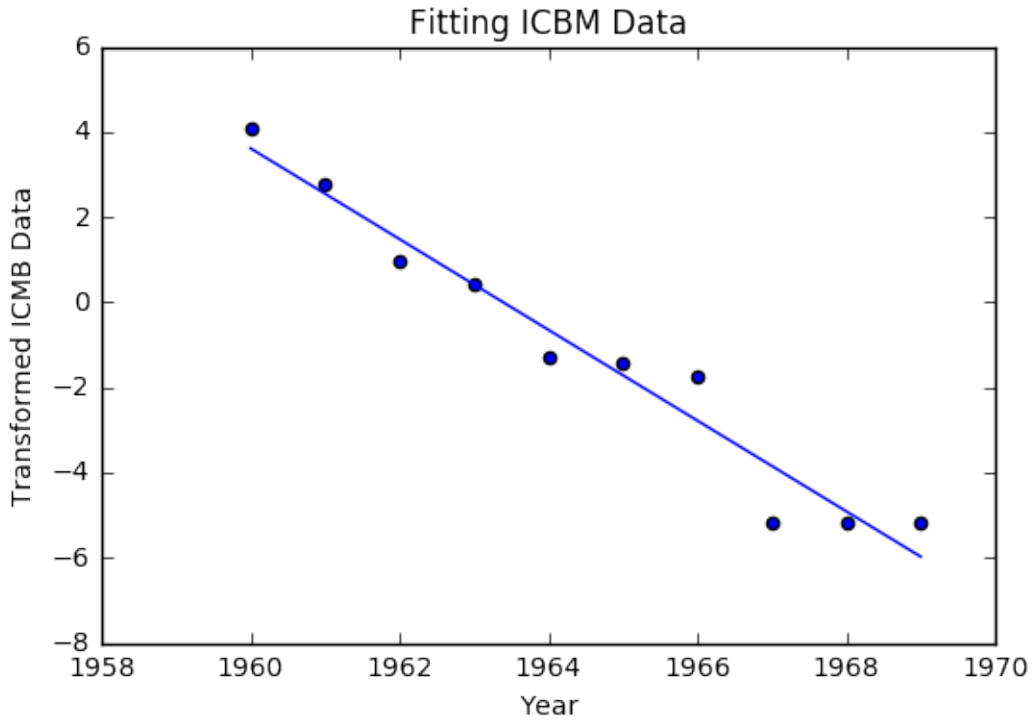
```
In [36]: t_icbm = [np.log((1060 - i)/i) for i in icbm]
```

```
In [37]: plt.scatter(year, t_icbm)
```

```
Out[37]: <matplotlib.collections.PathCollection at 0x111183cf8>
```




```
In [38]: b, a = np.polyfit(year, t_icbm, 1)

In [39]: a, b

Out[39]: (2091.7866057847828, -1.0653944179379069)

In [40]: def l(x):
             return b*x + a

         l(1960), l(1969)

Out[40]: (3.6135466264850038, -5.9750031349558412)

In [41]: fit = [l(i) for i in year]

In [42]: plt.scatter(year, t_icbm)
         plt.plot(year, fit)
         plt.title("Fitting ICBM Data")
         plt.xlabel("Year")
         plt.ylabel("Transformed ICMB Data")

Out[42]: <matplotlib.text.Text at 0x1111ed320>
```



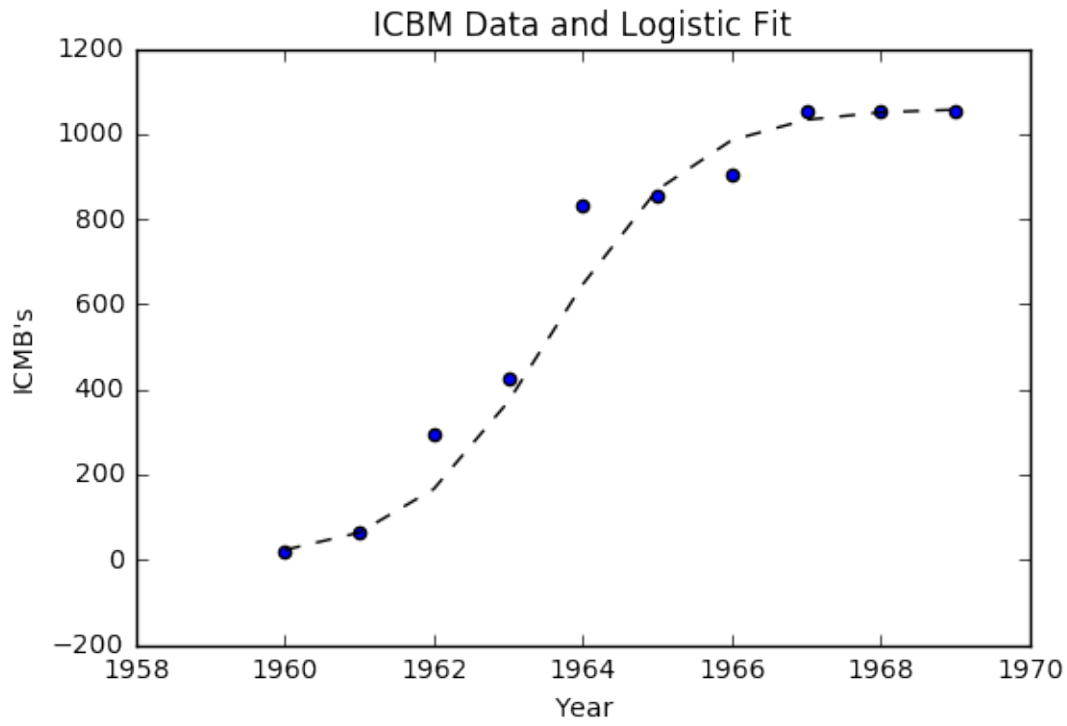
Much like the last example, we can return everything to its original form with the exponential. We arrive at the equation

$$y = \frac{1060}{1 + e^{2092 - 1.0654x}}$$

```
In [43]: def y(x):
             return 1060/(1 + np.exp(2092 - 1.0654*x))

         o_fit = [y(i) for i in year]
```

```
In [44]: plt.scatter(year, icbm)
         plt.plot(year, o_fit, '--k')
         plt.title("ICBM Data and Logistic Fit")
         plt.xlabel("Year")
         plt.ylabel("ICMB's")
```

```
Out[44]: <matplotlib.text.Text at 0x1111c0d30>
```



Machine Learning Example

In [46]:
```
import matplotlib.pyplot as plt
import numpy as np
from sklearn import datasets, linear_model
from sklearn.metrics import mean_squared_error, r2_score

# Load the diabetes dataset
diabetes = datasets.load_diabetes()
# Use only one feature
diabetes_X = diabetes.data[:, np.newaxis, 2]
```

In [61]:
```
diabetes_X[:5]
```

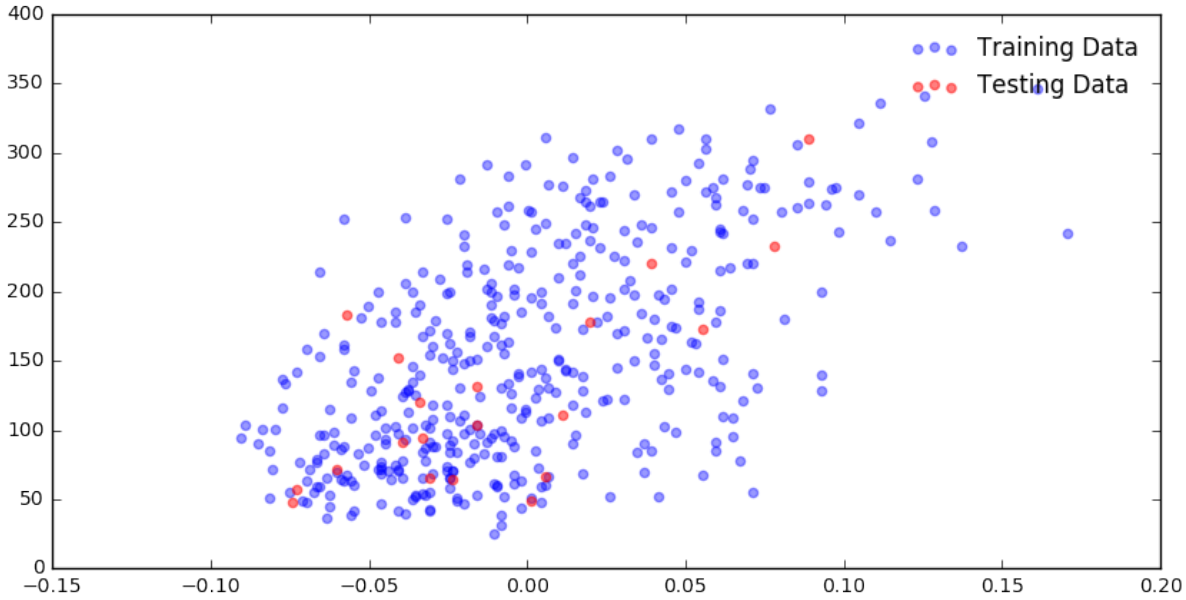
Out[61]:
```
array([[ 0.06169621],
       [-0.05147406],
       [ 0.04445121],
       [-0.01159501],
       [-0.03638469]])
```

In [64]:
```
datasets?
```

In [53]:
```
# Split the data into training/testing sets
diabetes_X_train = diabetes_X[:-20]
diabetes_X_test = diabetes_X[-20:]

# Split the targets into training/testing sets
diabetes_y_train = diabetes.target[:-20]
diabetes_y_test = diabetes.target[-20:]
```

In [60]:
```
plt.figure(figsize = (10,5))
plt.scatter(diabetes_X_train, diabetes_y_train, color = 'blue', alpha = 0.4, label = "Training Da
plt.scatter(diabetes_X_test, diabetes_y_test, color = 'red', alpha = 0.5, label = "Testing Data")
plt.legend(frameon = False)
```

Out[60]:
```
<matplotlib.legend.Legend at 0x1123e16a0>
```



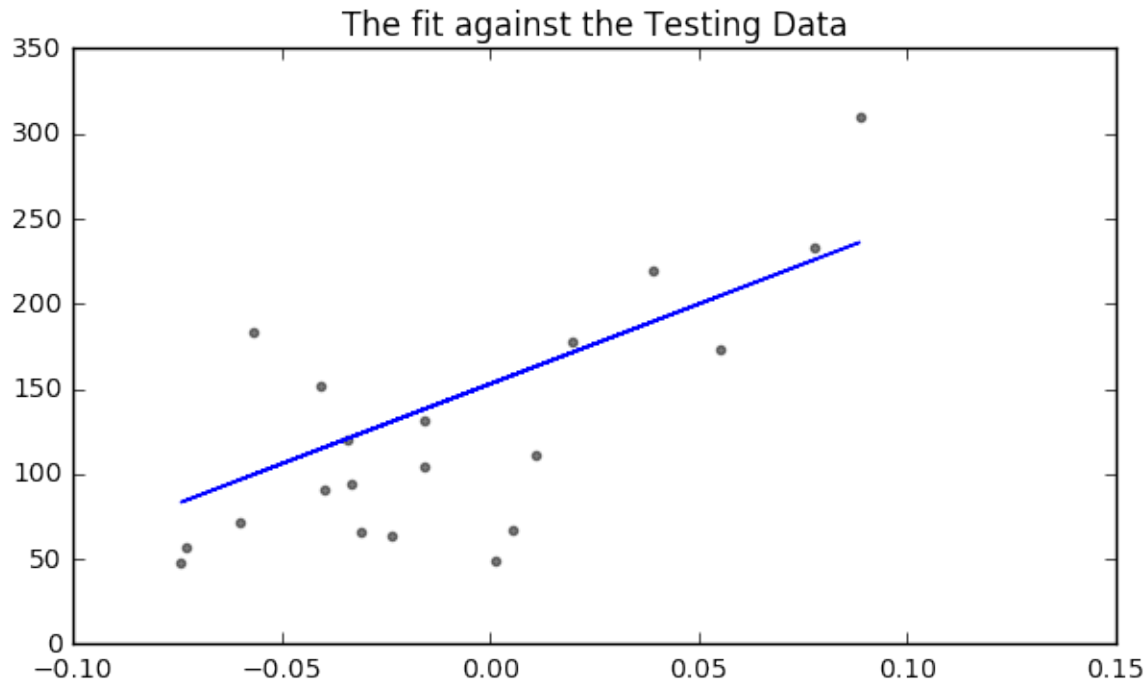
In [65]:
```
# Create linear regression object
regr = linear_model.LinearRegression()

# Train the model using the training sets
regr.fit(diabetes_X_train, diabetes_y_train)

# Make predictions using the testing set
diabetes_y_pred = regr.predict(diabetes_X_test)
```

In [66]:
```
# The coefficients
print('Coefficients: \n', regr.coef_)
# The mean squared error
print("Mean squared error: %.2f"
      % mean_squared_error(diabetes_y_test, diabetes_y_pred))
# Explained variance score: 1 is perfect prediction
print('Variance score: %.2f' % r2_score(diabetes_y_test, diabetes_y_pred))
```

```
Coefficients:
 [ 938.23786125]
Mean squared error: 2548.07
Variance score: 0.47
```

In [75]:
```
# Plot outputs
plt.figure(figsize = (7,4))
plt.scatter(diabetes_X_test, diabetes_y_test,  color='black', alpha = 0.5, s = 9)
plt.plot(diabetes_X_test, diabetes_y_pred, color='blue')
plt.title("The fit against the Testing Data")
```

Out[75]:
```
<matplotlib.text.Text at 0x112c6b748>
```



Problems

In [123]:
```
from sklearn.datasets import load_iris
iris = load_iris()
data = iris.data
column_names = iris.feature_names
```

In [134]:
```
print(iris.DESCR)
```

```
Iris Plants Database

Notes
-----
Data Set Characteristics:
    :Number of Instances: 150 (50 in each of three classes)
    :Number of Attributes: 4 numeric, predictive attributes and the class
    :Attribute Information:
        - sepal length in cm
        - sepal width in cm
        - petal length in cm
        - petal width in cm
        - class:
                - Iris-Setosa
                - Iris-Versicolour
                - Iris-Virginica
    :Summary Statistics:

    ============== ==== ==== ======= ===== ====================
                    Min  Max   Mean    SD   Class Correlation
    ============== ==== ==== ======= ===== ====================
    sepal length:   4.3  7.9   5.84   0.83    0.7826
    sepal width:    2.0  4.4   3.05   0.43   -0.4194
    petal length:   1.0  6.9   3.76   1.76    0.9490  (high!)
    petal width:    0.1  2.5   1.20  0.76     0.9565  (high!)
    ============== ==== ==== ======= ===== ====================
```

```
    :Missing Attribute Values: None
    :Class Distribution: 33.3% for each of 3 classes.
    :Creator: R.A. Fisher
    :Donor: Michael Marshall (MARSHALL%PLU@io.arc.nasa.gov)
    :Date: July, 1988

This is a copy of UCI ML iris datasets.
http://archive.ics.uci.edu/ml/datasets/Iris

The famous Iris database, first used by Sir R.A Fisher

This is perhaps the best known database to be found in the
pattern recognition literature.  Fisher's paper is a classic in the field and
is referenced frequently to this day.  (See Duda & Hart, for example.)  The
data set contains 3 classes of 50 instances each, where each class refers to a
type of iris plant.  One class is linearly separable from the other 2; the
latter are NOT linearly separable from each other.

References
----------
   - Fisher,R.A. "The use of multiple measurements in taxonomic problems"
     Annual Eugenics, 7, Part II, 179-188 (1936); also in "Contributions to
     Mathematical Statistics" (John Wiley, NY, 1950).
   - Duda,R.O., & Hart,P.E. (1973) Pattern Classification and Scene Analysis.
     (Q327.D83) John Wiley & Sons.  ISBN 0-471-22361-1.  See page 218.
   - Dasarathy, B.V. (1980) "Nosing Around the Neighborhood: A New System
     Structure and Classification Rule for Recognition in Partially Exposed
     Environments".  IEEE Transactions on Pattern Analysis and Machine
     Intelligence, Vol. PAMI-2, No. 1, 67-71.
   - Gates, G.W. (1972) "The Reduced Nearest Neighbor Rule".  IEEE Transactions
     on Information Theory, May 1972, 431-433.
   - See also: 1988 MLC Proceedings, 54-64.  Cheeseman et al"s AUTOCLASS II
     conceptual clustering system finds 3 classes in the data.
   - Many, many more ...
```

```
In [125]: data
```

```
Out[125]: array([[ 5.1,  3.5,  1.4,  0.2],
                 [ 4.9,  3. ,  1.4,  0.2],
                 [ 4.7,  3.2,  1.3,  0.2],
                 [ 4.6,  3.1,  1.5,  0.2],
                 [ 5. ,  3.6,  1.4,  0.2],
                 [ 5.4,  3.9,  1.7,  0.4],
                 [ 4.6,  3.4,  1.4,  0.3],
                 [ 5. ,  3.4,  1.5,  0.2],
                 [ 4.4,  2.9,  1.4,  0.2],
                 [ 4.9,  3.1,  1.5,  0.1],
                 [ 5.4,  3.7,  1.5,  0.2],
                 [ 4.8,  3.4,  1.6,  0.2],
                 [ 4.8,  3. ,  1.4,  0.1],
                 [ 4.3,  3. ,  1.1,  0.1],
                 [ 5.8,  4. ,  1.2,  0.2],
                 [ 5.7,  4.4,  1.5,  0.4],
                 [ 5.4,  3.9,  1.3,  0.4],
                 [ 5.1,  3.5,  1.4,  0.3],
                 [ 5.7,  3.8,  1.7,  0.3],
                 [ 5.1,  3.8,  1.5,  0.3],
                 [ 5.4,  3.4,  1.7,  0.2],
                 [ 5.1,  3.7,  1.5,  0.4],
                 [ 4.6,  3.6,  1. ,  0.2],
```

```
[ 5.1,  3.3,  1.7,  0.5],
[ 4.8,  3.4,  1.9,  0.2],
[ 5. ,  3. ,  1.6,  0.2],
[ 5. ,  3.4,  1.6,  0.4],
[ 5.2,  3.5,  1.5,  0.2],
[ 5.2,  3.4,  1.4,  0.2],
[ 4.7,  3.2,  1.6,  0.2],
[ 4.8,  3.1,  1.6,  0.2],
[ 5.4,  3.4,  1.5,  0.4],
[ 5.2,  4.1,  1.5,  0.1],
[ 5.5,  4.2,  1.4,  0.2],
[ 4.9,  3.1,  1.5,  0.1],
[ 5. ,  3.2,  1.2,  0.2],
[ 5.5,  3.5,  1.3,  0.2],
[ 4.9,  3.1,  1.5,  0.1],
[ 4.4,  3. ,  1.3,  0.2],
[ 5.1,  3.4,  1.5,  0.2],
[ 5. ,  3.5,  1.3,  0.3],
[ 4.5,  2.3,  1.3,  0.3],
[ 4.4,  3.2,  1.3,  0.2],
[ 5. ,  3.5,  1.6,  0.6],
[ 5.1,  3.8,  1.9,  0.4],
[ 4.8,  3. ,  1.4,  0.3],
[ 5.1,  3.8,  1.6,  0.2],
[ 4.6,  3.2,  1.4,  0.2],
[ 5.3,  3.7,  1.5,  0.2],
[ 5. ,  3.3,  1.4,  0.2],
[ 7. ,  3.2,  4.7,  1.4],
[ 6.4,  3.2,  4.5,  1.5],
[ 6.9,  3.1,  4.9,  1.5],
[ 5.5,  2.3,  4. ,  1.3],
[ 6.5,  2.8,  4.6,  1.5],
[ 5.7,  2.8,  4.5,  1.3],
[ 6.3,  3.3,  4.7,  1.6],
[ 4.9,  2.4,  3.3,  1. ],
[ 6.6,  2.9,  4.6,  1.3],
[ 5.2,  2.7,  3.9,  1.4],
[ 5. ,  2. ,  3.5,  1. ],
[ 5.9,  3. ,  4.2,  1.5],
[ 6. ,  2.2,  4. ,  1. ],
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[ 5.6,  2.9,  3.6,  1.3],
[ 6.7,  3.1,  4.4,  1.4],
[ 5.6,  3. ,  4.5,  1.5],
[ 5.8,  2.7,  4.1,  1. ],
[ 6.2,  2.2,  4.5,  1.5],
[ 5.6,  2.5,  3.9,  1.1],
[ 5.9,  3.2,  4.8,  1.8],
[ 6.1,  2.8,  4. ,  1.3],
[ 6.3,  2.5,  4.9,  1.5],
[ 6.1,  2.8,  4.7,  1.2],
[ 6.4,  2.9,  4.3,  1.3],
[ 6.6,  3. ,  4.4,  1.4],
[ 6.8,  2.8,  4.8,  1.4],
[ 6.7,  3. ,  5. ,  1.7],
[ 6. ,  2.9,  4.5,  1.5],
[ 5.7,  2.6,  3.5,  1. ],
[ 5.5,  2.4,  3.8,  1.1],
[ 5.5,  2.4,  3.7,  1. ],
[ 5.8,  2.7,  3.9,  1.2],
```

```
[ 6. ,  2.7,  5.1,  1.6],
[ 5.4,  3. ,  4.5,  1.5],
[ 6. ,  3.4,  4.5,  1.6],
[ 6.7,  3.1,  4.7,  1.5],
[ 6.3,  2.3,  4.4,  1.3],
[ 5.6,  3. ,  4.1,  1.3],
[ 5.5,  2.5,  4. ,  1.3],
[ 5.5,  2.6,  4.4,  1.2],
[ 6.1,  3. ,  4.6,  1.4],
[ 5.8,  2.6,  4. ,  1.2],
[ 5. ,  2.3,  3.3,  1. ],
[ 5.6,  2.7,  4.2,  1.3],
[ 5.7,  3. ,  4.2,  1.2],
[ 5.7,  2.9,  4.2,  1.3],
[ 6.2,  2.9,  4.3,  1.3],
[ 5.1,  2.5,  3. ,  1.1],
[ 5.7,  2.8,  4.1,  1.3],
[ 6.3,  3.3,  6. ,  2.5],
[ 5.8,  2.7,  5.1,  1.9],
[ 7.1,  3. ,  5.9,  2.1],
[ 6.3,  2.9,  5.6,  1.8],
[ 6.5,  3. ,  5.8,  2.2],
[ 7.6,  3. ,  6.6,  2.1],
[ 4.9,  2.5,  4.5,  1.7],
[ 7.3,  2.9,  6.3,  1.8],
[ 6.7,  2.5,  5.8,  1.8],
[ 7.2,  3.6,  6.1,  2.5],
[ 6.5,  3.2,  5.1,  2. ],
[ 6.4,  2.7,  5.3,  1.9],
[ 6.8,  3. ,  5.5,  2.1],
[ 5.7,  2.5,  5. ,  2. ],
[ 5.8,  2.8,  5.1,  2.4],
[ 6.4,  3.2,  5.3,  2.3],
[ 6.5,  3. ,  5.5,  1.8],
[ 7.7,  3.8,  6.7,  2.2],
[ 7.7,  2.6,  6.9,  2.3],
[ 6. ,  2.2,  5. ,  1.5],
[ 6.9,  3.2,  5.7,  2.3],
[ 5.6,  2.8,  4.9,  2. ],
[ 7.7,  2.8,  6.7,  2. ],
[ 6.3,  2.7,  4.9,  1.8],
[ 6.7,  3.3,  5.7,  2.1],
[ 7.2,  3.2,  6. ,  1.8],
[ 6.2,  2.8,  4.8,  1.8],
[ 6.1,  3. ,  4.9,  1.8],
[ 6.4,  2.8,  5.6,  2.1],
[ 7.2,  3. ,  5.8,  1.6],
[ 7.4,  2.8,  6.1,  1.9],
[ 7.9,  3.8,  6.4,  2. ],
[ 6.4,  2.8,  5.6,  2.2],
[ 6.3,  2.8,  5.1,  1.5],
[ 6.1,  2.6,  5.6,  1.4],
[ 7.7,  3. ,  6.1,  2.3],
[ 6.3,  3.4,  5.6,  2.4],
[ 6.4,  3.1,  5.5,  1.8],
[ 6. ,  3. ,  4.8,  1.8],
[ 6.9,  3.1,  5.4,  2.1],
[ 6.7,  3.1,  5.6,  2.4],
[ 6.9,  3.1,  5.1,  2.3],
[ 5.8,  2.7,  5.1,  1.9],
```

```
       [ 6.8,  3.2,  5.9,  2.3],
       [ 6.7,  3.3,  5.7,  2.5],
       [ 6.7,  3. ,  5.2,  2.3],
       [ 6.3,  2.5,  5. ,  1.9],
       [ 6.5,  3. ,  5.2,  2. ],
       [ 6.2,  3.4,  5.4,  2.3],
       [ 5.9,  3. ,  5.1,  1.8]])
```

In [126]:
```
df = pd.DataFrame(data)
```

In [127]:
```
df.columns = column_names
```

In [128]:
```
df.head()
```

Out[128]:

	sepal length (cm)	sepal width (cm)	petal length (cm)	petal width (cm)
0	5.1	3.5	1.4	0.2
1	4.9	3.0	1.4	0.2
2	4.7	3.2	1.3	0.2
3	4.6	3.1	1.5	0.2
4	5.0	3.6	1.4	0.2

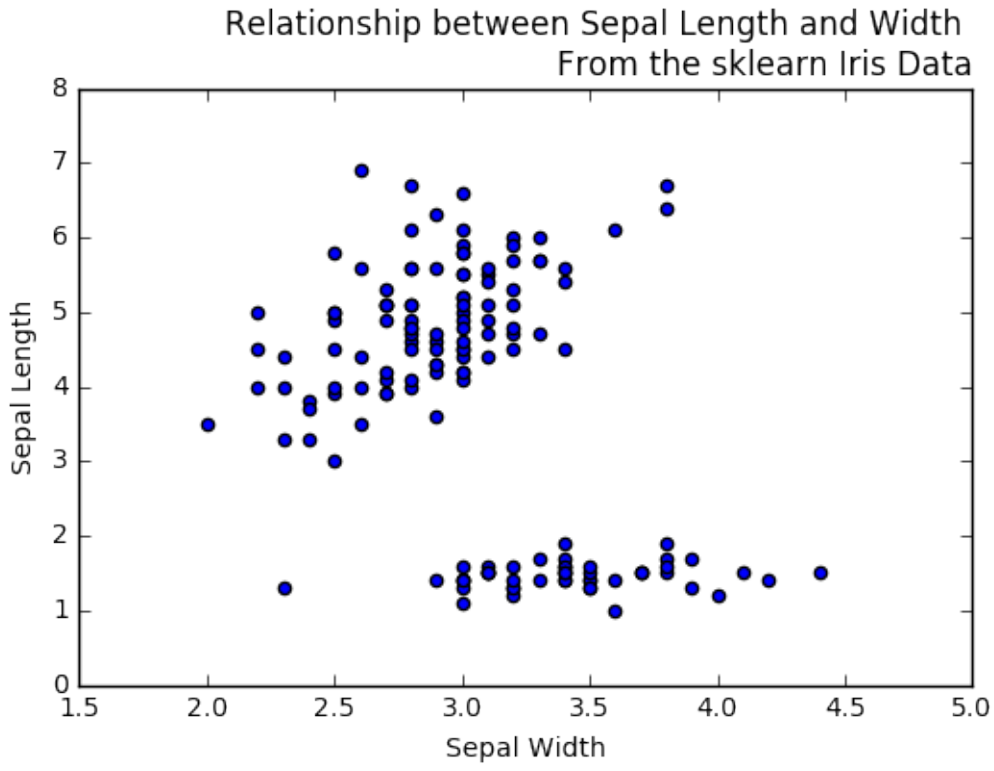
In [129]:
```
df.iloc[:, [1]].head()
```

Out[129]:

	sepal width (cm)
0	3.5
1	3.0
2	3.2
3	3.1
4	3.6

In [130]:
```
plt.figure()
plt.scatter(df.iloc[:,1], df.iloc[:,2])
plt.xlabel("Sepal Width")
plt.ylabel("Sepal Length")
plt.title("Relationship between Sepal Length and Width \nFrom the sklearn Iris Data", loc = 'rig
```

Out[130]:
```
<matplotlib.text.Text at 0x11b63c080>
```



In [131]:
```
df.describe()
```

Out[131]:
```
       sepal length (cm)  sepal width (cm)  petal length (cm)  \
count         150.000000        150.000000         150.000000
mean            5.843333          3.054000           3.758667
std             0.828066          0.433594           1.764420
min             4.300000          2.000000           1.000000
25%             5.100000          2.800000           1.600000
50%             5.800000          3.000000           4.350000
75%             6.400000          3.300000           5.100000
max             7.900000          4.400000           6.900000

       petal width (cm)
count        150.000000
mean           1.198667
std            0.763161
min            0.100000
25%            0.300000
50%            1.300000
75%            1.800000
max            2.500000
```

In [132]:
```
from sklearn.datasets import load_boston
boston = load_boston()
bdata = boston.data
bcolumn_names = boston.feature_names
```

In [133]:
```
print(boston.DESCR)
```

```
Boston House Prices dataset

Notes
------
Data Set Characteristics:

    :Number of Instances: 506

    :Number of Attributes: 13 numeric/categorical predictive

    :Median Value (attribute 14) is usually the target

    :Attribute Information (in order):
        - CRIM     per capita crime rate by town
        - ZN       proportion of residential land zoned for lots over 25,000 sq.ft.
        - INDUS    proportion of non-retail business acres per town
        - CHAS     Charles River dummy variable (= 1 if tract bounds river; 0 otherwise)
        - NOX      nitric oxides concentration (parts per 10 million)
        - RM       average number of rooms per dwelling
        - AGE      proportion of owner-occupied units built prior to 1940
        - DIS      weighted distances to five Boston employment centres
        - RAD      index of accessibility to radial highways
        - TAX      full-value property-tax rate per $10,000
        - PTRATIO  pupil-teacher ratio by town
        - B        1000(Bk - 0.63)^2 where Bk is the proportion of blacks by town
        - LSTAT    % lower status of the population
        - MEDV     Median value of owner-occupied homes in $1000's

    :Missing Attribute Values: None

    :Creator: Harrison, D. and Rubinfeld, D.L.

This is a copy of UCI ML housing dataset.
```

```
http://archive.ics.uci.edu/ml/datasets/Housing


This dataset was taken from the StatLib library which is maintained at Carnegie Mellon University.

The Boston house-price data of Harrison, D. and Rubinfeld, D.L. 'Hedonic
prices and the demand for clean air', J. Environ. Economics & Management,
vol.5, 81-102, 1978.   Used in Belsley, Kuh & Welsch, 'Regression diagnostics
...', Wiley, 1980.   N.B. Various transformations are used in the table on
pages 244-261 of the latter.

The Boston house-price data has been used in many machine learning papers that address regression
problems.

**References**

   - Belsley, Kuh & Welsch, 'Regression diagnostics: Identifying Influential Data and Sources of Collinear
   - Quinlan,R. (1993). Combining Instance-Based and Model-Based Learning. In Proceedings on the Tenth Int
   - many more! (see http://archive.ics.uci.edu/ml/datasets/Housing)
```

```
In [117]: df2 = pd.DataFrame(bdata)
```

```
In [118]: df2.columns = bcolumn_names
```

```
In [119]: df2.head()
```

```
Out[119]:      CRIM    ZN  INDUS  CHAS    NOX     RM   AGE     DIS  RAD    TAX  \
          0  0.00632  18.0   2.31   0.0  0.538  6.575  65.2  4.0900  1.0  296.0
          1  0.02731   0.0   7.07   0.0  0.469  6.421  78.9  4.9671  2.0  242.0
          2  0.02729   0.0   7.07   0.0  0.469  7.185  61.1  4.9671  2.0  242.0
          3  0.03237   0.0   2.18   0.0  0.458  6.998  45.8  6.0622  3.0  222.0
          4  0.06905   0.0   2.18   0.0  0.458  7.147  54.2  6.0622  3.0  222.0

             PTRATIO       B  LSTAT
          0     15.3  396.90   4.98
          1     17.8  396.90   9.14
          2     17.8  392.83   4.03
          3     18.7  394.63   2.94
          4     18.7  396.90   5.33
```

```
In [1]: %matplotlib inline
        import matplotlib.pyplot as plt
        import numpy as np
        import sympy as sy
        from scipy.integrate import odeint
```

1.4 Anti-derivatives, Inverse Tangents, and Differential Equations

Many of the important early problems surrounding the mathematical circles of Leibniz and Newton involved what we now call differential equations and what they called the inverse tangent problem. For example, we can write the equation

$$\frac{dy}{dx} = 2x + 3$$

This equation is giving us information about the tangent lines of a relationship. Whatever this relationship is, we know the slope of the tangent line at any point (x, y) is simply $2x + 3$.

What we want, is to determine a relationship that satisfies these requirements. The example above is easy enough to see that a possible solution is

$$y = x^2 + 3x$$

```
In [2]: def dy(x):
            return 2*x + 3

        x = sy.Symbol('x')
        anti_d = sy.integrate(2*x+3, x)
        sy.pprint(anti_d)
```

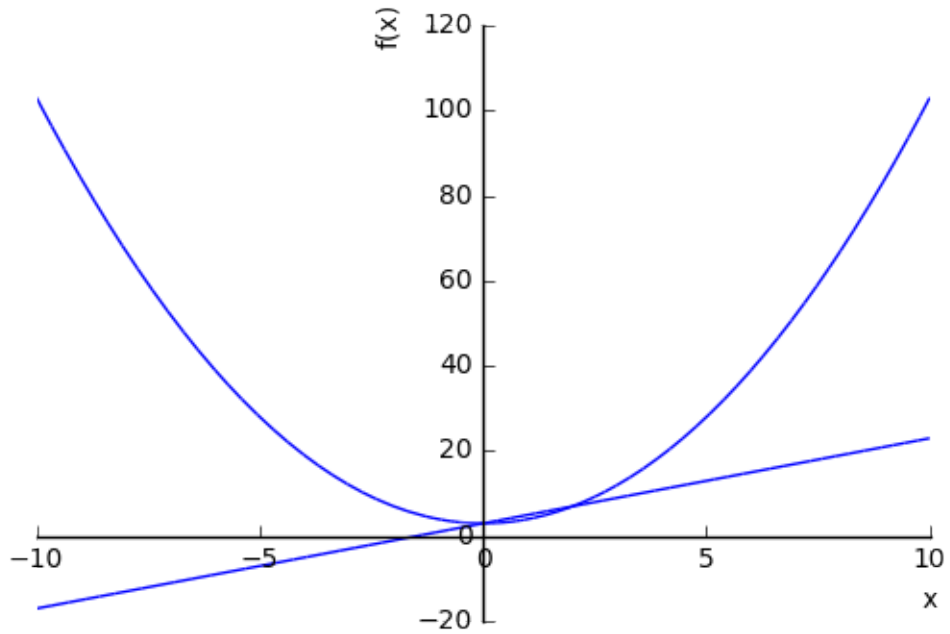
```
 2
x  + 3x
```

We can check the work by seeing that the derivative of our antiderivative is what we started with.

```
In [3]: sy.diff(anti_d, x)
```

```
Out[3]: 2*x + 3
```

```
In [4]: sy.plot(x**2 + 3, 2*x + 3)
```




```
Out[4]: <sympy.plotting.plot.Plot at 0x1154f7cf8>
```

Familiar Examples

We can frame a few familiar examples in terms of differential equations. In doing so, we express a relationship involving a rate of change. The above example could be interpreted as asking for a relationship whose rate of change is a linear function with a rate of change of 1 unit of x. We can similarly consider some sequence whose differences are changing in a linear fashion.

Compound Interest

In compound interest, we know that each compounding period t results in multiplying the amount in the account A by the interest rate r. In terms of differential equations, we would have that the rate of change of the investment A is proportional to the amount present.

$$\frac{dA}{dt} = rA$$

We can use SymPy to find a solution to this, which should verify our existing knowledge of the solution.

```
In [5]: t, A, r, P = sy.symbols('t A r P')

In [6]: A = sy.Function("A")(t)
        A_ = sy.Derivative(A, t)
        P = sy.Symbol('P')

In [7]: b = sy.Eq(A_, r*A)

In [8]: soln = sy.dsolve(b)
        sy.pprint(soln)

           rt
A(t) = C
```

We find that we are given the solution with some other constant C_1. We will examine this in more detail in just a second, but for now let's visit some familiar examples to get used to solving the differential equations in SymPy.

Introductory Examples

Solve the differential equation with Sympy:

- $\frac{dy}{dx} = y^2$
- $4x^2\frac{d^2y}{dx^2} + y = 0$
- $y' + 2xy^2 = 0$
- $y' = y + 2e^{-x}$

Families of AntiDerivatives

One of the issues with antiderivatives is that they are not unique. For example, our first differential equation actually has an infinite number of solutions of the form

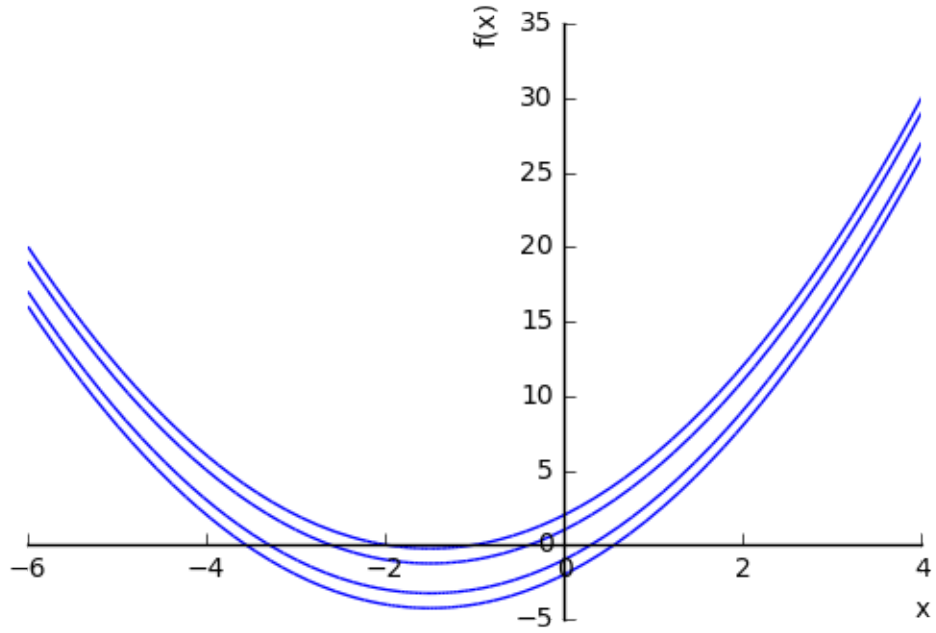
$$y = \frac{x^2}{2} + 2x + c$$

where c is any real number. We can see this by checking some values of c.

```
In [9]: y2 = x**2 + 3*x - 1
        y3 = x**2 + 3*x -2
        y4 = x**2 + 3*x + 1
        y5 = x**2 + 3*x + 2

In [10]: dy2 = sy.diff(y2, x)
         dy3 = sy.diff(y3, x)
         dy4 = sy.diff(y4, x)
         dy5 = sy.diff(y5, x)

In [11]: dy2, dy3, dy4, dy5

Out[11]: (2*x + 3, 2*x + 3, 2*x + 3, 2*x + 3)
```

If we look at the plots of these different solutions, we see some similarities and differences.

```
In [12]: sy.plot(y2, y3, y4, y5, (x, -6,4))
```



Out[12]: <sympy.plotting.plot.Plot at 0x1180c6e48>

In a similar manner, explore alternative values for the constants of integration and plot the solutions to your earlier problems.

Initial Value Problems

Now, we will suppose that we know a certain point on the curve that the solution passes through. For example, or early example began with the differential equation

$$\frac{dy}{dx} = 2x + 3$$

We found that the solutions to this equation take the form

$$y = x^2 + 3x + C$$

where C is some constant.

```
In [13]: y = sy.Function("y")(x)
         dy = sy.Derivative(y, x)
         diff_eq = sy.Eq(dy, 2*x + 3)
         soln = sy.dsolve(diff_eq)
         sy.pprint(soln)
```

```
             2
y(x) = C + x  + 3x
```

We may know some additional information however. Suppose we knew that the solution that we wanted passed through the point $(1,2)$. This would mean that if we substitute for each of these values, we should get a true statement. We can find a value for C_1 then, that would make this true.

```
In [14]: c = sy.Symbol('c')
         initial_value = sy.Eq(2, 1**2 + 3*1 + c)
         sy.solve(initial_value, c)
```

Out[14]: [-2]

Hence, we have that

$$y = x^2 + 3x - 2$$

We find a more general solution for our compound interest problem in a similar manner where we have the initial investment P at time $t = 0$.

```
In [15]: P = sy.Symbol('P')
         initial_value = sy.Eq(P, c*sy.exp(r*0))
         sy.solve(initial_value, c)
```

```
Out[15]: [P]
```

Hence, we have our solution as

$$y = Pe^{rt}$$

Motion as Differential Equation

Before, we investigated the relationships between *displacement*, *velocity*, and *acceleration*. In the language of differential equations, we can phrase this as

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

where we have

$x = f(t)$

for the motion of some particle along the x-axis.

```
In [16]: a = sy.Symbol('a')
         v = sy.Function("v")(t)
         dv = sy.Derivative(v, t)
         diff_eq = sy.Eq(dv, a)
         soln = sy.dsolve(diff_eq)
         sy.pprint(soln)
```

```
v(t) = C + at
```

```
In [17]: v0 = sy.Symbol('v0')
         initial_value = sy.Eq(c + a*0, v0)
         sy.solve(initial_value, c)
```

```
Out[17]: [v0]
```

```
In [18]: x = sy.Function("x")(t)
         dx = sy.Derivative(x, t)
         diff_eq = sy.Eq(dx, v0 + a*t)
         soln2 = sy.dsolve(diff_eq)
         sy.pprint(soln2)
```

```
              2
             at
x(t) = C +   + tv
             2
```

Thus we have

$$v(t) = at + v_0$$

and

$$x(t) = \frac{1}{2}at^2 + v_0t + x_0$$

Problems

Fintd the position function $x(t)$ for some particle given the information about the constant acceleration a, the initial velocity v_0, and initial position x_0.

- $a(t) = 40, v_0 = 5, x_0 = 10$
- $a(t) = -2, v_0 = 20, x_0 = 34$
- $a(t) = 3t, v_0 = 2, x_0 = 4$

Force and Gravitational Acceleration

One of the accomplishments of Newton's *Principia* was to set down laws of motion that were developed by assuming a universal force of gravity acting on all bodies. The acceleration due to the Earth's gravity is approximated by 9.8 m/s^2. Further, we can define the *weight* of a body as the force exerted by gravity on a body. Newton's second law states that the Force on any body is equal to its mass times acceleration. We call the unit of force required to move impart an acceleration of 1 m/s^2 to a mass of 1 kg a Newton.

Thus, we have that $W = mg$.

Returning to our problems with motion, we note that we view the action on bodies on Earth as an acceleration due to gravity g, hence we have

$$a = \frac{dv}{dt}$$

$$v(t) = -gt + v_0$$

$$y(t) = \frac{-1}{2}gt^2 + v_0 t + y_0$$

and can use these equations to solve some basic problems dealing with bodies in motion on Earth. For example, suppose we throw a ball straight up from the ground with an initial velocity $v_0 = 90$ ft/sec, hence it reaches its maximum height when the velocity is zero, or

$$v(t) = -32t + 96 = 0$$

or

$$t = 3$$

so we have a maximum height of

$$y(3) = -1/2(32)(3^2) + 96(3) + 0 = 144ft$$

Problems

- A projectile is fired straight up from the top of a building 30m high, with initial velocity of 40m/s. Find its maximum height above the ground, when it passes the top of the building and its time in the air.
- A bomb is dropped from a helicopter hovering at an altitude of 800 feet above the ground. From the ground directly beneath the helicopter, a projectile is fired straight upward toward the bomb, exacly 2 seconds after the bomb is released. With waht initial velocity should the projectile be fired, in order to hit the bomb at nan altitude of exacly 400 feet?

```
In [18]: %matplotlib inline
         import matplotlib.pyplot as plt
         import numpy as np
         import sympy as sy
         from math import atan2
```

1.5 Visualizing Differential Equations

Returning to the example from our last notebook, we had the differential equation

$$\frac{dy}{dx} = 2x + 3$$

Interpreting this geometrically we are looking at a relationship where the slope of the tangent line at some point (x, y) is given by $2x + 3$. For example, at any point where $x = 1$, we have the slope of the tangent line equal to $2(1) + 3 = 5$. We can draw a small short line at each of these points to represent this behavior. Similarly, we could repeat this for a small number of x and y values. Most important, is that we understand we are not finding a single line solution, but rather representing tangent lines of a family of functions just like our initial general solutions to the ODE.

In [2]:
```
def dy_dx(x):
    return 2*x + 3
```

In [3]:
```
x = np.arange(-3,4, 1)
y = np.arange(-3,4, 1)
X,Y = np.meshgrid(x, y)
```

In [4]:
```
X
```

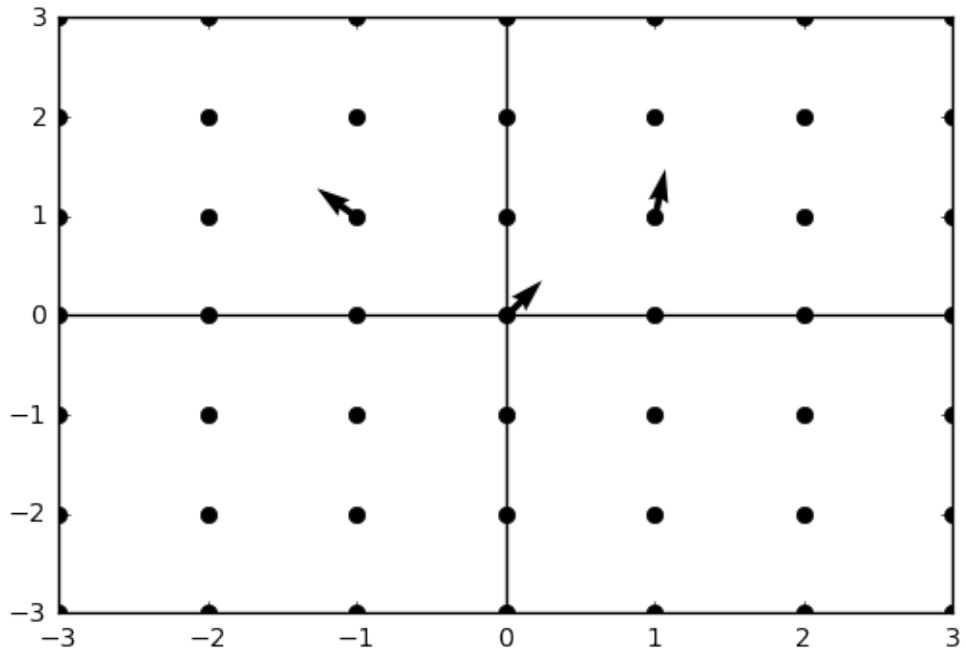
Out[4]:
```
array([[-3, -2, -1,  0,  1,  2,  3],
       [-3, -2, -1,  0,  1,  2,  3],
       [-3, -2, -1,  0,  1,  2,  3],
       [-3, -2, -1,  0,  1,  2,  3],
       [-3, -2, -1,  0,  1,  2,  3],
       [-3, -2, -1,  0,  1,  2,  3],
       [-3, -2, -1,  0,  1,  2,  3]])
```

In [5]:
```
Y
```

Out[5]:
```
array([[-3, -3, -3, -3, -3, -3, -3],
       [-2, -2, -2, -2, -2, -2, -2],
       [-1, -1, -1, -1, -1, -1, -1],
       [ 0,  0,  0,  0,  0,  0,  0],
       [ 1,  1,  1,  1,  1,  1,  1],
       [ 2,  2,  2,  2,  2,  2,  2],
       [ 3,  3,  3,  3,  3,  3,  3]])
```

In [122]:
```
fig = plt.figure()
plt.plot(X,Y, 'o', color = 'black', markersize = 6)
plt.axhline(color = 'black')
plt.axvline(color = 'black')
ax = fig.add_subplot(111)
ax.quiver(0,0,2,2)
ax.quiver(1,1,1.1,2*(1.1) + 3)
ax.quiver(-1,1,-1.1, 2*(-1.1) + 3)
```

Out[122]:
```
<matplotlib.quiver.Quiver at 0x11fcd9160>
```



In [127]:
```
atan2(dy_dx(2), 1)
```

Out[127]:
```
1.4288992721907328
```

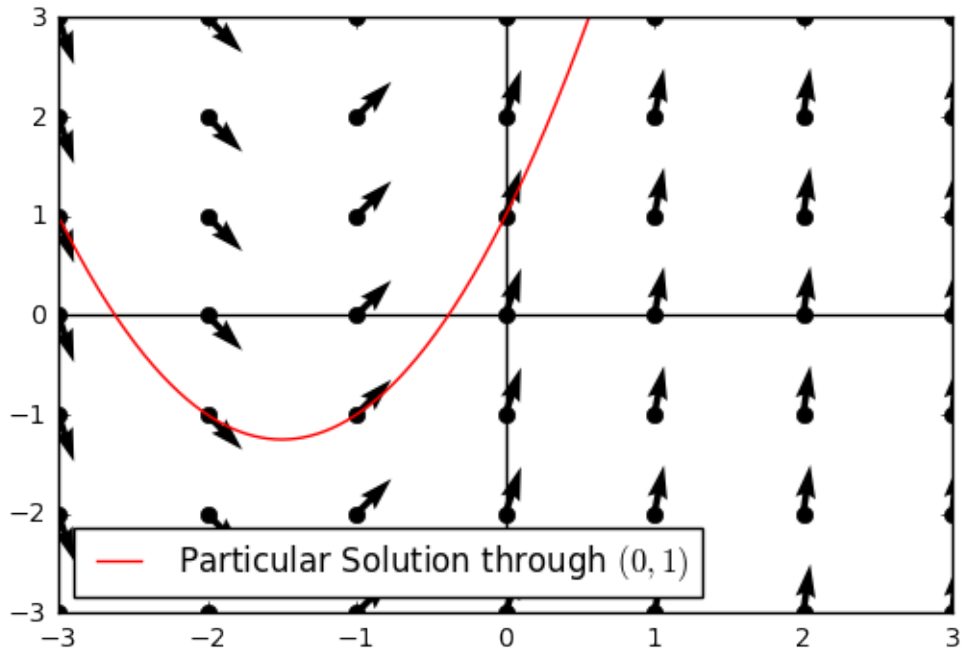
In [134]:
```
theta = [atan2(dy_dx(i), 1) for i in x]
```

In [140]:
```
theta
```

Out[140]:
```
[-1.2490457723982544,
 -0.7853981633974483,
 0.7853981633974483,
 1.2490457723982544,
 1.373400766945016,
 1.4288992721907328,
 1.460139105621001]
```

In [147]:
```
plt.quiver(X,Y, np.cos(theta), np.sin(theta))
plt.axhline(color = 'black')
plt.axvline(color = 'black')
plt.plot(X, Y, 'o', color = 'black')
x2 = np.linspace(-3,3,1000)
plt.plot(x2, x2**2 + 3*x2 + 1, '-k', color = 'red', label = 'Particular Solution through $(0,1)$
plt.xlim(-3,3)
plt.ylim(-3,3)
plt.legend(loc = 'best')
```

Out[147]:
```
<matplotlib.legend.Legend at 0x120e02438>
```




```
In [1]: %matplotlib inline
        import matplotlib.pyplot as plt
        import numpy as np
        import sympy as sy
```

1.6 Approximating Values

In this notebook, we approximate solutions to differential equations. This is an important idea, as most differential equations are not easily solved with classical techniques. Our solutions with be numerical values that approximate the exact solutions to the differential equation. Given the initial value problem:

$$\frac{dy}{dx} = 2x + 3 \quad y(0) = 1$$

we want to determine other values of the function near $x = 0$. To start, we want to consider how to approximate the following values **without** solving the differential equation, as we already know this is $y = x^2 + 3x + 1$.

- $y(0.1)$
- $y(0.2)$
- $y(0.3)$
- $y(1.1)$

For $y(0.1)$, consider using the tangent line to approximate this value. We have a line with slope $2(0.1) + 3 = 3.2$ through the point $(0, 1)$. We can write an equation for this line as

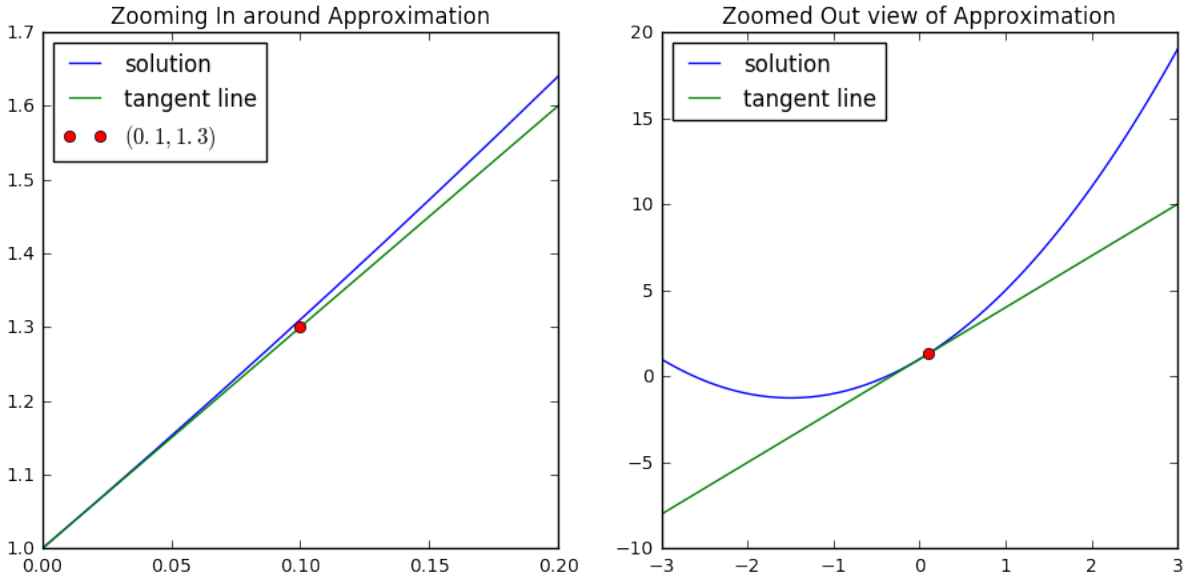
$$y = 3.2(x - 0) + 1$$

and investigate this line and the solution around $x = 0.1$

```
In [2]: plt.figure(figsize = (11,5))
        plt.subplot(121)
        x = np.linspace(0,0.2,10)
        plt.plot(x, x**2 + 3*x + 1, label = 'solution')
        plt.plot(x, 3*x + 1, label = 'tangent line')
```

```
plt.plot(0.1,1.3,'o', label = '$(0.1, 1.3)$')
plt.legend(loc = 'best')
plt.title('Zooming In around Approximation')

plt.subplot(122)
x = np.linspace(-3,3,100)
plt.plot(x, x**2 + 3*x + 1, label = 'solution')
plt.plot(x, 3*x + 1, label = 'tangent line')
plt.plot(0.1,1.3,'o')
plt.legend(loc = 'best')
plt.title('Zoomed Out view of Approximation')
```

Out[2]: <matplotlib.text.Text at 0x11312a320>



As we notice in the graph on the right, while the tangent line serves as a nice approximation in the near vicinity of our initial value, as we move further away it diverges from the solution.

Euler's Method

Also, recognize that the differential equation suggests an alternative equation for the tangent line at different x values. At $x = 1.1$ the tangent line has a different slope than when $x = 1$. We can adjust every 0.1 and write a new equation to traverse for a short period of time. Here, we started at $x = 0$, moved along the tangent line for 0.1 units and now write the equation for the tangent line at $x = 0.1$.

$$y_0 = 1$$

$$tan_1 = (2(0) + 3)(x - 0) + 1 = 3x + 1$$

When $x = 0.1$, we have $3(0.1) + 1 = 1.3$, so

$$y_1 = 1.3$$

We repeat this process at $(0.1, 1.3)$ in order to find y_2.

$$y_2 = (2(0.1) + 3)(0.2 - 0.1) + 1.3 = 1.62$$

to find y_3 or an approximation for $y(0.3)$, we simply repeat

$$y_3 = (2(x_2) + 3)(x_3 - x_2) + y_2 = 1.96$$

```
In [3]: (2*(0.1) + 3)*(0.2 - 0.1)+ 1.3
```

```
Out[3]: 1.62
```

```
In [4]: (2*(0.2) + 3)*(0.3 - 0.2)+ 1.62
```

```
Out[4]: 1.96
```

Perhaps you recognize a more general pattern. We are evaluating the derivative at each h step, multiplying this by h and adding this to the previous approximation to get our new approximation. Let's generate more terms and visualize our solutions. Formally, we can define a relationship between successive y terms as follows, where $x_i = h * i$ and $f = \frac{dy}{dx}$:

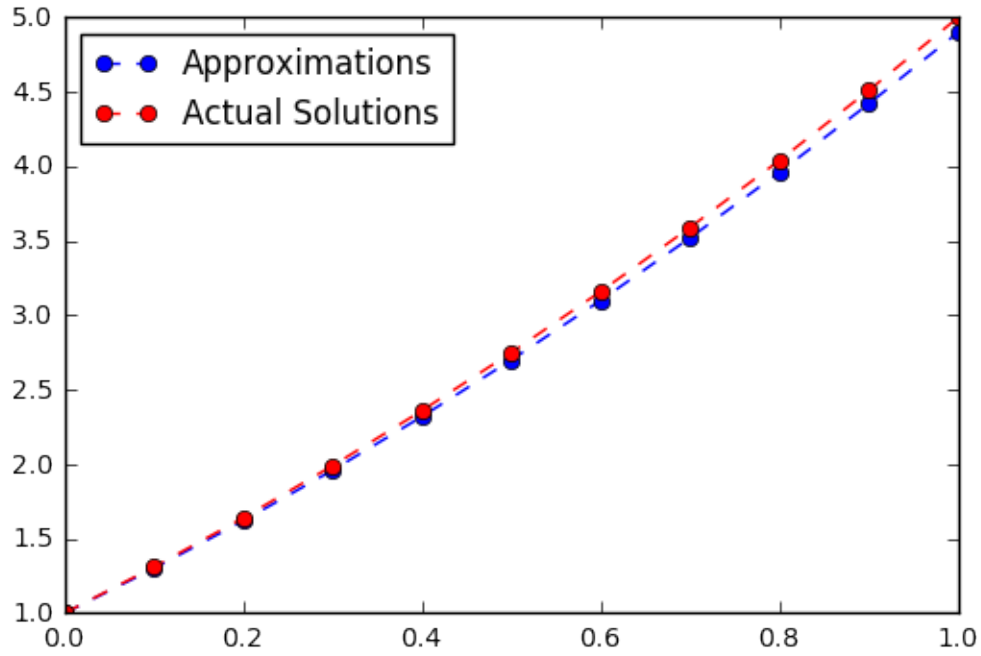
$$y_{i+1} = hf(x_i) + y_i$$

```
In [5]: def df(x):
            return 2*x+3

        h = 0.1
        y = [1]
        for i in range(10):
            next = df(i*h)*h + y[i]
            y.append(next)

        y
```

```
Out[5]: [1,
         1.3,
         1.62,
         1.9600000000000002,
         2.3200000000000003,
         2.7,
         3.1,
         3.52,
         3.96,
         4.42,
         4.9]
```

```
In [6]: x = np.linspace(0,1.0,11)
        plt.plot(x,y,'--o', label = 'Approximations')
        plt.plot(x, x**2 + 3*x + 1, '--o', color = 'red', label = 'Actual Solutions')
        plt.legend(loc = 'best')
```

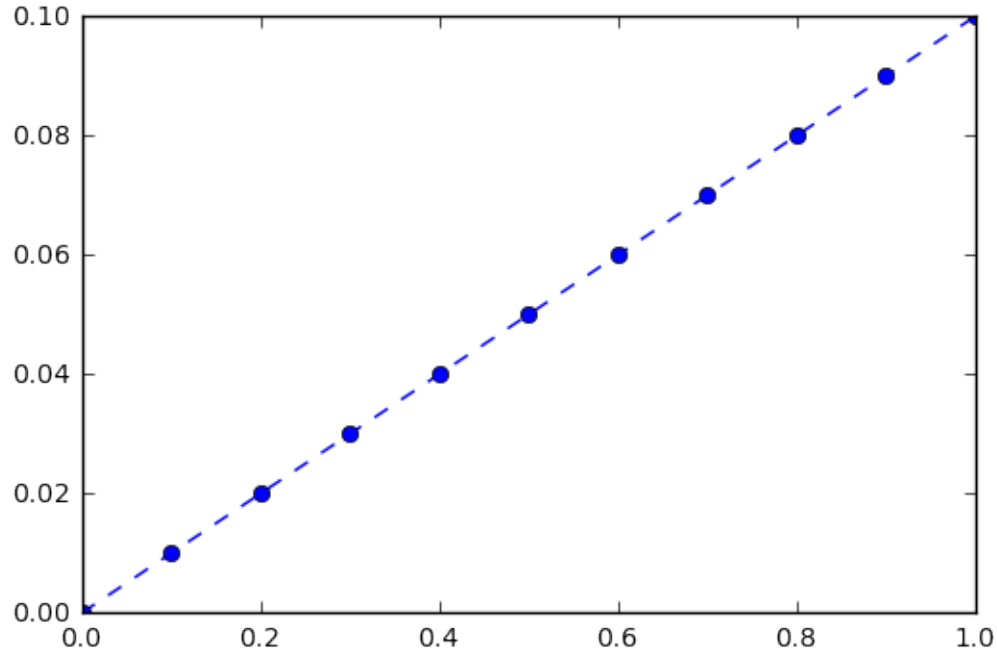
```
Out[6]: <matplotlib.legend.Legend at 0x113916dd8>
```




```
In [7]: solutions = [(i**2 + 3*i + 1) for i in x]
        error = [(solutions[i] - y[i]) for i in range(len(x))]
```

```
In [8]: plt.plot(x, error, '--o')
```

```
Out[8]: [<matplotlib.lines.Line2D at 0x113b1e3c8>]
```

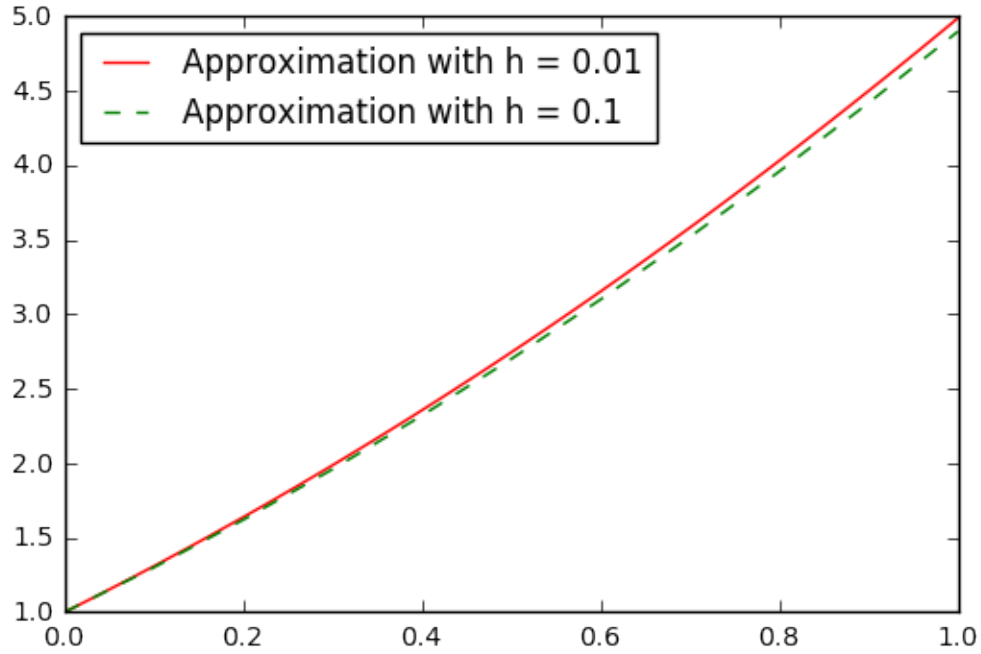



```
In [9]: h = 0.01
```

```
In [10]: y2 = [1]
         for i in range(100):
             next = df(i*h)*h + y2[i]
             y2.append(next)

         x2 = [i for i in np.linspace(0,1.0,101)]
```

```
plt.plot(x2, y2, '-k', color = 'red', label = 'Approximation with h = 0.01')
plt.plot(x, y, '--k', color = 'green', label = 'Approximation with h = 0.1')
plt.legend(loc = 'best')
```

Out[10]: <matplotlib.legend.Legend at 0x113c40b38>

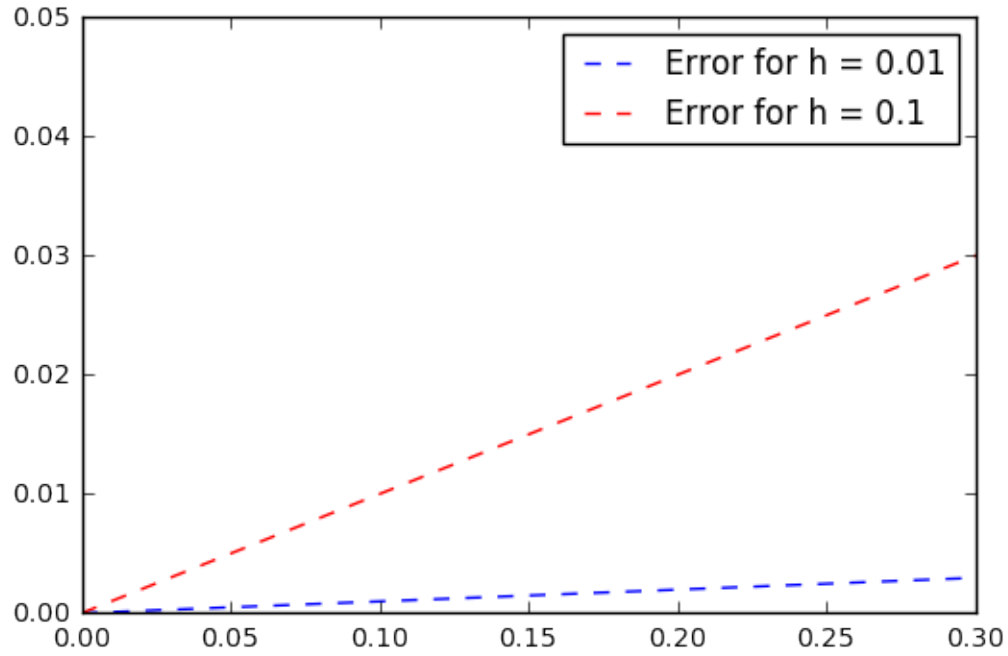


You may not feel this is a big difference, however if we examine the error in these approximations you should notice the divergence in the values depending on the size of h.

In [11]:
```
solutions2 = [(i**2 + 3*i + 1) for i in np.linspace(0,1,101)]
error2 = [(solutions2[i] - y2[i]) for i in range(100)]
x2 = [i for i in np.linspace(0,1,100)]
```

In [12]:
```
plt.plot(x2, error2, '--', color = 'blue', label = 'Error for h = 0.01')
plt.plot(x, error, '--k', color = 'red', label = 'Error for h = 0.1')
plt.legend()
plt.xlim(0,0.3)
plt.ylim(0,0.05)
```

Out[12]: (0, 0.05)



Population Models

Another familiar differential equation is the exponential growth/decay model. Here, we have a rate of change of a population A at some time t for growth rate r as

$$\frac{dP}{dt} = rP(t)$$

Also, suppose we have the initial condition that at $t = 0$ the population is $P(0) = 100$, with $r = 2$. Use Euler's method to approximate solutions at time $t = .1, .2, .4, 1.1$ with $h = 0.1, 0.01$, and 0.001. Compare the errors in these approximations to the exact solution to the differential equation.

Populations II

Now suppose we have the differential equation below that models a biological population in time given in hours. Determine an appropriate step size to provide reasonable approximations to the differential equation.

$$\frac{dP}{dt} = 0.003P(t)(2 + 5\sin(\frac{\pi}{12}t)) \quad P(0) = 45000$$

Comparing Exact and Approximates

As mentioned earlier, we cannot always find a function who has a derivative expressed by the differential equation and instead need to use an approximation to find values of a solution. Below, two of the three involve precise solutions. Determine approximations for each, and those that you can find an exact solution using any method please do so.

- $\frac{dy}{dt} = \cos(t) - y\tan(t) \quad y(0) = 0$
- $\frac{dy}{dt} = t\cos(yt) \quad y(0) = 0$
- $\frac{dy}{dt} = t^2 \quad y(0) = 0$

```
In [1]: %matplotlib notebook
        import matplotlib.pyplot as plt
        import numpy as np
        import sympy as sy
        from scipy.integrate import odeint
        size = (12, 9)
        from ipywidgets import interact, widgets, fixed
        from math import atan2
```

1.7 Population Models

A simple example of a model involving a differential equation could be the basic additive population growth model. Here, suppose we have a constant rate of change k. As a differential equation we would have:

$$\frac{dP}{dt} = k$$

We are familiar with the solution

$$P(t) = kt + c$$

In this notebook, we want to add complexity to this model and explore the results. Obviously, assuming a constant rate of change in a population model is a fairly naive assumption. Regardless, let's quickly revisit our methods for visualizing and solving a differential equation expressed in this form.

```
In [2]: x, k, t, P = sy.symbols('x k t P')
```

```
In [3]: def dp_dt(x,k):
            return k
```

```
In [4]: sy.integrate(dp_dt(x,k), t)
```

```
Out[4]: k*t
```

```
In [5]: x = np.linspace(0,10,11)
        theta = [atan2(dp_dt(i, 1.1), 1) for i in x]
```

```
In [6]: x = np.linspace(0,10,11)
        y = np.linspace(0,10,11)
        X, Y = np.meshgrid(x, y)
        plt.figure()
        plt.plot(X, Y, 'o', color = 'black', alpha = 0.6)
        plt.quiver(X,Y, np.cos(theta), np.sin(theta))
        plt.plot(x, 1.1*x + 4, '-k', color = 'red', linewidth = 4, label = 'Solution Curve for $k = 1.1$ a
        plt.ylim(0,10)
        plt.legend(loc = 'best')
```

```
<IPython.core.display.Javascript object>
```

```
<IPython.core.display.HTML object>
```

```
Out[6]: <matplotlib.legend.Legend at 0x119402ef0>
```

Exponential Growth

Last notebook, we saw the basic population model expressed as a differential equation. Here, we should recall our earlier work with discrete sequences and the passage to the limit that determines the change from ΔP to dP. We will rely heavily on this idea when modeling with differential equations. For the simple exponential population model, as a differential equation we have

$$\frac{dP}{dt} = rP$$

whereas in the discrete case we have

$$\frac{\Delta P}{\Delta t} = rP \quad or \quad \Delta P = rP\Delta t$$

Returning to a basic example, suppose we know a population has size $P = 100$ at time $t = 0$. The population grows at a 0.24% growth rate. If we consider this in terms of our abover relationship as a way to approximate solutions every $\Delta t = 0.1$, we get the following populations p_i.

$$p_0 = 100$$

$$p_1 = 100 + 0.0024(100)(0.1) \quad \text{or} \quad p_0(1+r)\Delta t$$

Problem: A certain city had a population of 25,000 and a growth rate of $r = 1.8 exponentially at a constant rate, what population can its city planners expect in the year 2000?$

```
In [7]: r = 0.018
        p0 = 25000
        dt = 1

        P = [p0]
        for i in range(10):
            next = P[i]*(1 + r)*dt
            P.append(next)
```

```
In [8]: P
```

```
Out[8]: [25000,
         25450.0,
         25908.100000000002,
         26374.4458,
         26849.185824400003,
         27332.471169239205,
         27824.455650285512,
         28325.295851990653,
         28835.151177326487,
         29354.183898518364,
         29882.559208691695]
```

```
In [9]: plt.figure()
        plt.plot(np.linspace(1,10,11), P, '-o')
```

```
<IPython.core.display.Javascript object>
```

```
<IPython.core.display.HTML object>
```

```
Out[9]: [<matplotlib.lines.Line2D at 0x11946fd68>]
```

We are interested in the effect of varying some of the parameters of the differential equation. Here, we really only have two, the inital population and the growth rate. Let's just vary the growth rate and describe the change in the resulting solutions.

```
In [10]: def exp_grow(p0, r):
             P = [p0]
             for i in range(10):
                 next = p0*r**i
                 P.append(next)
             plt.figure()
             plt.plot(P, '-o')

         interact(exp_grow, r=(1.00,2.0,0.01), p0 = (100,500,25));
```

The Logistic Differential Equation

The logistic equation was introduced by Pierre-Francois Verhulst in a discussion on the United States population growth. In the middle of the 19th century, the United States was still expanding, and the population had followed exponential growth. This could not continue however, as "the difficulty of finding good land has begun to make itself felt."

"the population tends to grow in geometric progression while the production of food follows a more or less arithmetic progression"–Verhulst 1845

This translates mathematically to the rate of births against those of deaths as a linear rate of change. We can examine this through an example.

Suppose we have a fish population that will stop growing at a population of 50,000. We also know that when the population is 10,000, the population doubles. Accordingly, if we plot population against growth rate we have two points $(10,000, 2)$ and $(50,000, 1)$. We assume this is linear, and have the equation:

$$r = 2 - 0.000025(p - 10,000)$$

```
In [11]: plt.figure(figsize=(8,6))
         p = np.linspace(0,74,2000)
         r = 2.25 - .025*p
         plt.plot(p,r)
         plt.plot(50,1,'ro')
         plt.plot(10,2,'ro')
         plt.xlim(0,72)
         plt.ylabel('Growth Rate r')
         plt.xlabel('Population in Thousands')
         plt.title('A linear change in growth rate')
```

```
<IPython.core.display.Javascript object>
```

```
<IPython.core.display.HTML object>
```

```
Out[11]: <matplotlib.text.Text at 0x119502b38>
```

Now we want to find some values. We can approximate these values using the earlier general population model. In this situation we end up with:

$$p_{n+1} = r * p_n$$

$$p_{n+1} = (2.25 - .000025p_n)p_n$$

We will define a function to generate successive values of the sequence.

```
In [12]: def logistic_example(N=100, x0 = 1000):
             x = np.zeros(N)
             x[0] = x0
             for n in range(N-1):
                 x[n+1] = (2.25 - 0.000025*x[n])*x[n]
             return x

         x = logistic_example(N=100, x0 = 1000)
```

```
In [13]: n = np.linspace(0,10,10)
         plt.figure()
         plt.plot(n,x[:10],marker='o')
         plt.xlabel('Year')
         plt.ylabel('Population')
         plt.title('Logistic Fish Population')
```

```
<IPython.core.display.Javascript object>
```

P. F. VERHULST.

```
<IPython.core.display.HTML object>
```

```
Out[13]: <matplotlib.text.Text at 0x119431828>
```

```
In [14]: def plot_logistic(i,x):
             plt.figure(figsize=(9,6))
             plt.plot(x[:i],'-ok', linewidth=2)
             plt.ylim(0,60000); plt.xlim(0,len(x))
             plt.xlabel('r'); plt.ylabel('x')
             plt.xlabel('Year')
             plt.ylabel('Population')
             plt.title('Logistic Fish Population')
```

```
In [15]: x = logistic_example(N=20)
         plot_logistic(len(x), x)
```

```
<IPython.core.display.Javascript object>
```

```
<IPython.core.display.HTML object>
```

Experimenting with different rates

In general, we are dealing with the recurrence relationship

$$x_{n+1} = rx_n(1 - x_n)$$

We've seen what happened in the fish population with a certain rate. Now we would like to see some other behavior by playing with values of r.

```
In [16]: def logistic_example2(r=1.0, N=100, x0 = 0.6):
             x = np.zeros(N)
             x[0] = x0
             for n in range(N-1):
                 x[n+1] = r*x[n]*(1. - x[n])
             return x
```

```
In [17]: c = logistic_example2(r=1.0, N=30)
         print(c[:10])
```

```
[ 0.6         0.24        0.1824      0.14913024  0.12689041  0.11078923
  0.09851498  0.08880978  0.0809226   0.07437413]
```

```
In [18]: def plot_logistic2(i,x):
             plt.figure(figsize=(9,5))
             plt.plot(c[:i], '-ok', linewidth=2)
```

```
In [19]: interact(plot_logistic2, i = widgets.IntSlider(min=0, max=len(c), step=1, value=0), x=fixed(x));
```

```
/Users/NYCMath/anaconda/lib/python3.5/site-packages/traitlets/traitlets.py:567: FutureWarning: comparison
  silent = bool(old_value == new_value)
```

Changing r

We can see some different kinds of behavior depending on altering the parameters r and x_0. Three things may happen. Recall our equation:

$$x_{n+1} = x_n(1 - x_n)$$

1. The values of x_n get closer and closer to some limit value x_∞ as $n \to \infty$.
2. The values of x_n oscillate periodically among two or more values, repeating those values forever.
3. The values x_n get larger and larger without bound.

```
In [20]: x = logistic_example2(r=1.5,N=30,x0=1./10)
         def plot_logistic2(i,x):
             plt.figure(figsize=(9,5))
             plt.plot(x[:i], '-ok', linewidth=2)
         plot_logistic2(len(x),x)
```

```
<IPython.core.display.Javascript object>
```

```
<IPython.core.display.HTML object>
```

Exploration

Playing with the sliders below, answer the following:

1. For what values of r does $x_n \to 0$?
2. What happens for slightly bigger values of r?
3. Can you find values of r that generate periodic orbits (i.e., situation 2 from above list)?
4. How many different points are visited in the periodic case?
5. What happens for even bigger values of r?
6. Make sure to try the value $r = 3.83$. What do you observe?

```
In [21]: def logistic(r=1.0, N = 100, x0=0.2):
             x = np.zeros(N)
             x[0] = x0
             for n in range(N-1):
                 x[n+1] = r * x[n] * (1. - x[n])
             plt.figure(figsize=size)
             ax1 = plt.subplot2grid((1,8), (0,0), colspan=7)
             ax2 = plt.subplot2grid((1,8), (0,7), colspan=1)

             ax1.plot(x, '-ok', linewidth=2)
             ax1.set_ylim(0,1)
             n = int(round(N/5))
             ax2.plot([0]*n,x[-n:],'or',markersize=10,alpha=0.1)
             ax2.set_ylim(0,1)
             ax2.axis('off')
             ax1.set_xlabel('r'); ax1.set_ylabel('x')

         interact(logistic,r=(0,4,0.01),x0=(0.01,1,0.1));
```

```
In [22]: def bifurcation_diagram(r=(0.8,4),N=2000,k=2000,m=200,x0=0.2):
             """
                 r: Pair of numbers (rmin,rmax) indicating parameter range
                 k: Number of samples in r
                 N: Number of iterations per sequence
                 m: keep just the last m iterates
             """
             x = np.zeros((k,N))
             vals = np.zeros((k,m))
             rs = np.linspace(r[0],r[1],k)
             x[:,0] = x0
             for n in range(N-1):
                 x[:,n+1] = rs * x[:,n] * (1. - x[:,n])
             return rs, x[:,-m:]

         plotargs = {'markersize':0.5, 'alpha':0.4}
```

```
        rs, vals = bifurcation_diagram()
        plt.figure(figsize=size)
        plt.plot(rs,vals,'ok',**plotargs);
        plt.xlim(rs.min(),rs.max());
        plt.xlabel('r'); plt.ylabel('x');
```

```
<IPython.core.display.Javascript object>
```

```
<IPython.core.display.HTML object>
```

```
In [1]: %matplotlib inline
        import matplotlib.pyplot as plt
        import numpy as np
        import sympy as sy
        from ipywidgets import interact, widgets, fixed
```

1.8 Population Models

The simplest population model we examined was that of exponential growth. Here, we claimed a population that grows in direct proportion to itself can be represented both in terms of discrete time steps and a continuous differential equation model, given below.

$$p_{n+1} = rp_n \qquad \frac{dP}{dt} = kP$$

Additionally, we looked at a population model that considered the limiting behavior of decreased resources through the logistic equation. Again, we could describe this either based on a discrete model or a continuous model as show below.

$$p_{n+1} = m(L - p_n)p_n \qquad \frac{dP}{dt} = rP(1 - P)$$

This model was interesting because of some unexpected behavior that we saw in the form of chaos. Today, we extend our discussion to population dynamics where we investigate predator prey relationships. Given some population of prey $x(t)$ and some population of prey $y(t)$ we aim to investigate the changes in these populations over time depending on the values of certain population parameters much like we did with the logistic model. In general, we will be exploring variations on a theme. The general system of equations given by:

$$\frac{dx}{dt} = f_1(t, x, y) - f_2(t, x, y)$$

$$\frac{dy}{dt} = f_3(t, x, y) - f_4(t, x, y)$$

Can be understood as describing the change in populations based on the difference between the increase in the populations (f_1, f_3) and the decrease in the populations (f_2, f_4).

Lotka Voltera Model

An early example of a predator prey model is the Lotka Volterra model. We can follow Lotka's argument for the construction of the model. First, he assumed:

- In the absence of predators, prey should grow exponentially without bound
- In the absence of prey, predators population should decrease similarly

Thus, we have the following:

$$\frac{dx}{dt} = ax$$

$$\frac{dy}{dt} = -dy$$

where a, d are some constants. Next, Lotka called on the **Law of Mass Action** from chemistry. When a reaction occurs by mixing chemicals, the law of mass action states that the rate of the reaction is proportional to the product of the quantities of the reactants. Lotka argued that prey should decrease and that predators should increase at rates proportional to the product of numbers present. Hence, we would have:

$$\frac{dx}{dt} = ax - bxy$$

$$\frac{dy}{dt} = cxy - dy$$

For some other constants b and c.

```
In [2]: def predator_prey(a=2,b=1,c=1,d=5):
            # Initial populations
            rabbits = [10]
            foxes = [20]

            a = a/1000.
            b = b/10000.
            c = c/10000.
            d = d/1000.

            dt = 0.1

            N = 20000
            for i in range(N):
                R = rabbits[-1]
                F = foxes[-1]
                change_rabbits = dt * (a*R - b*R*F)
                change_foxes = dt * (c*R*F - d*F)
                rabbits.append(rabbits[-1] + change_rabbits)
                foxes.append(foxes[-1] + change_foxes)

            plt.figure(figsize=(12,6))
            plt.plot(np.arange(N+1),rabbits)
            plt.hold(True)
            plt.plot(np.arange(N+1),foxes)
            plt.ylim(0,200); plt.xlabel('Time'); plt.ylabel('Population')
            plt.legend(('Rabbits','Foxes'),loc='best');
```

Questions

Examine the following Lotka Volterra model:

$$\frac{dx}{dt} = 0.7x - 0.3xy$$

$$\frac{dy}{dt} = 0.08xy - 0.44y$$

with $x(0) = 4$ and $y(0) = 2$.

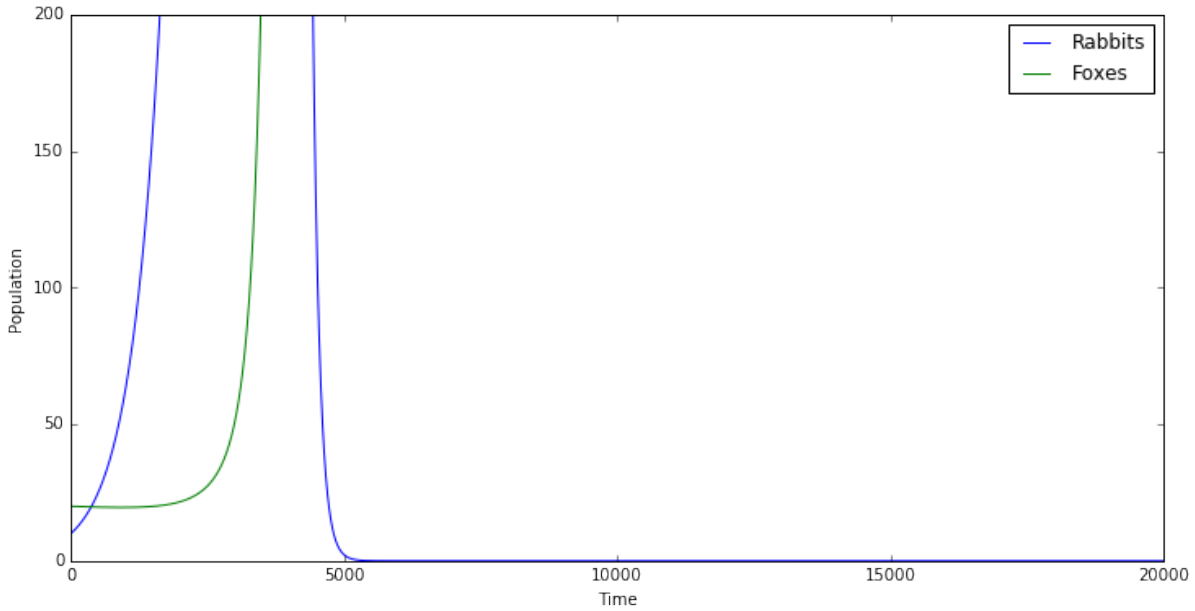
1. Experiment with changing the rate constants to answer the following questions:

- Describe the effect of increasing the prey birth rate. Is this what you expected to happen? Explain.
- Describe the effect of increasing the prey death rate. Is this what you expected to happen? Explain.

- The predator birth rate is currently much less than the prey birth rate. What effect does having the same birth rate for both species have on the solution?

2. Experiment with the initial populations for the predators and prey. Experiment with incorporating harvesting terms into both of the equations. What happens when you change the harvesting rate? Can you harvest and have the populations increase rather than decrease? Is this what you expected? Explain.

```
In [3]: interact(predator_prey,a=widgets.FloatSlider(min=0.01,max=30,step=1,value=0.7),
                               b=widgets.FloatSlider(min=0.01,max=10,step=1,value=0.3),
                               c=widgets.FloatSlider(min=0.01,max=10,step=1,value=0.08),
                               d=widgets.FloatSlider(min=0.01,max=30,step=1,value=0.44));
```



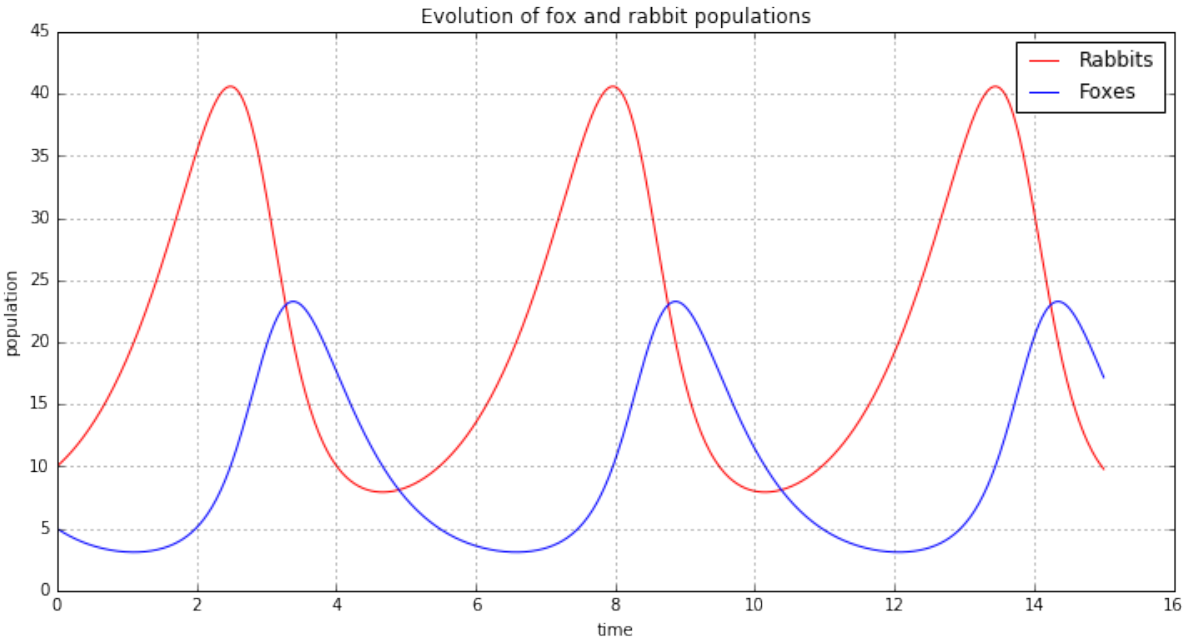
Now let's look at a particular example. We will set $a, b, c, d = 1, 0.1, 1.5$, and 0.75 respectively. We can use the scipy `integrate` function to solve the differential equations with the initial conditions as populations of 10 rabbits and 5 foxes.

```
In [4]: a = 1.
        b = 0.1
        c = 1.5
        d = 0.75
        def dX_dt(X, t=0):
            """ Return the growth rate of fox and rabbit populations. """
            return array([ a*X[0] -   b*X[0]*X[1] ,
                          -c*X[1] + d*b*X[0]*X[1] ])
```

```
In [5]: from scipy import integrate
        from numpy import *
        t = linspace(0, 15,  1000)              # time
        X0 = array([10, 5])                     # initials conditions: 10 rabbits and 5 foxes
        X = integrate.odeint(dX_dt, X0, t)
```

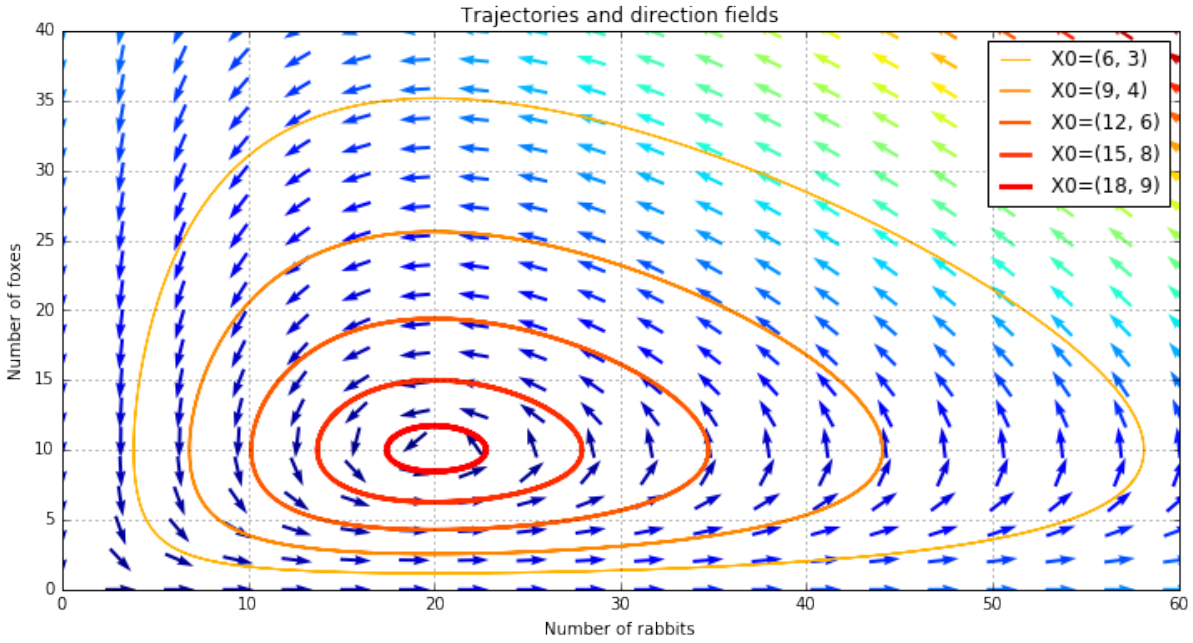
```
In [6]: rabbits, foxes = X.T
        f1 = plt.figure(figsize=(12,6))
        plt.plot(t, rabbits, 'r-', label='Rabbits')
        plt.plot(t, foxes  , 'b-', label='Foxes')
        plt.grid()
        plt.legend(loc='best')
        plt.xlabel('time')
```

```
plt.ylabel('population')
plt.title('Evolution of fox and rabbit populations')
f1.savefig('rabbits_and_foxes_1.png')
```



In [7]:
```
X_f0 = array([     0. ,  0.])
X_f1 = array([ c/(d*b), a/b])
all(dX_dt(X_f0) == zeros(2) ) and all(dX_dt(X_f1) == zeros(2))

values  = linspace(0.3, 0.9, 5)                          # position of X0 between X_f0 and X_f1
vcolors = plt.cm.autumn_r(linspace(0.3, 1., len(values)))  # colors for each trajectory

f2 = plt.figure(figsize=(12,6))

#-----------------------------------------------------------
# plot trajectories
for v, col in zip(values, vcolors):
    X0 = v * X_f1                               # starting point
    X = integrate.odeint( dX_dt, X0, t)         # we don't need infodict here
    plt.plot( X[:,0], X[:,1], lw=3.5*v, color=col, label='X0=(%.f, %.f)' % ( X0[0], X0[1]) )

#-----------------------------------------------------------
# define a grid and compute direction at each point
ymax = plt.ylim(ymin=0)[1]                        # get axis limits
xmax = plt.xlim(xmin=0)[1]
nb_points   = 20

x = linspace(0, xmax, nb_points)
y = linspace(0, ymax, nb_points)

X1 , Y1  = meshgrid(x, y)                       # create a grid
DX1, DY1 = dX_dt([X1, Y1])                      # compute growth rate on the gridt
M = (hypot(DX1, DY1))                           # Norm of the growth rate
M[ M == 0] = 1.                                 # Avoid zero division errors
DX1 /= M                                        # Normalize each arrows
DY1 /= M

#-----------------------------------------------------------
# Drow direction fields, using matplotlib 's quiver function
# I choose to plot normalized arrows and to use colors to give information on
```

```
# the growth speed
plt.title('Trajectories and direction fields')
Q = plt.quiver(X1, Y1, DX1, DY1, M, pivot='mid', cmap=plt.cm.jet)
plt.xlabel('Number of rabbits')
plt.ylabel('Number of foxes')
plt.legend()
plt.grid()
plt.xlim(0, xmax)
plt.ylim(0, ymax)
f2.savefig('rabbits_and_foxes_2.png')
```




```
In [1]: %matplotlib inline
        import matplotlib.pyplot as plt
        import numpy as np
        import pandas as pd
        import sympy as sy
```

1.9 Discrete Dynamical Systems

Below, we will see a few examples of discrete dynamical systems. We will start with the example of medication dosage. Suppose we have some amount of medication in our system upon taking a dose one day later. We would have a fraction plus the dose in our system at this time. Below, the example is first examined numerically by iterating a number of terms of the equation. Symbolically, this means suppose we start with $\alpha = 0.8$, and $n = 1$:

$$n_2 = 0.8 \times n_1 + 1$$

$$n_3 = 0.8 \times n_2 + 1$$

$$\vdots$$

In python, we can index how many iterations we want to carry out with the `range()` function, and embed the iteration in a `for` loop as follows.

```
In [2]: n = 1
        for i in range(1,10):
            n = 0.8*n + 1
            print(n)
```

```
1.8
2.4400000000000004
2.9520000000000004
3.3616000000000006
3.6892800000000006
3.9514240000000007
4.161139200000001
4.328911360000001
4.4631290880000005
```

We can look at some additional terms and see that these are heading towards 5.

```
In [3]: n = 1
        for i in range(1,50):
            n = 0.8*n + 1
            print(n)
```

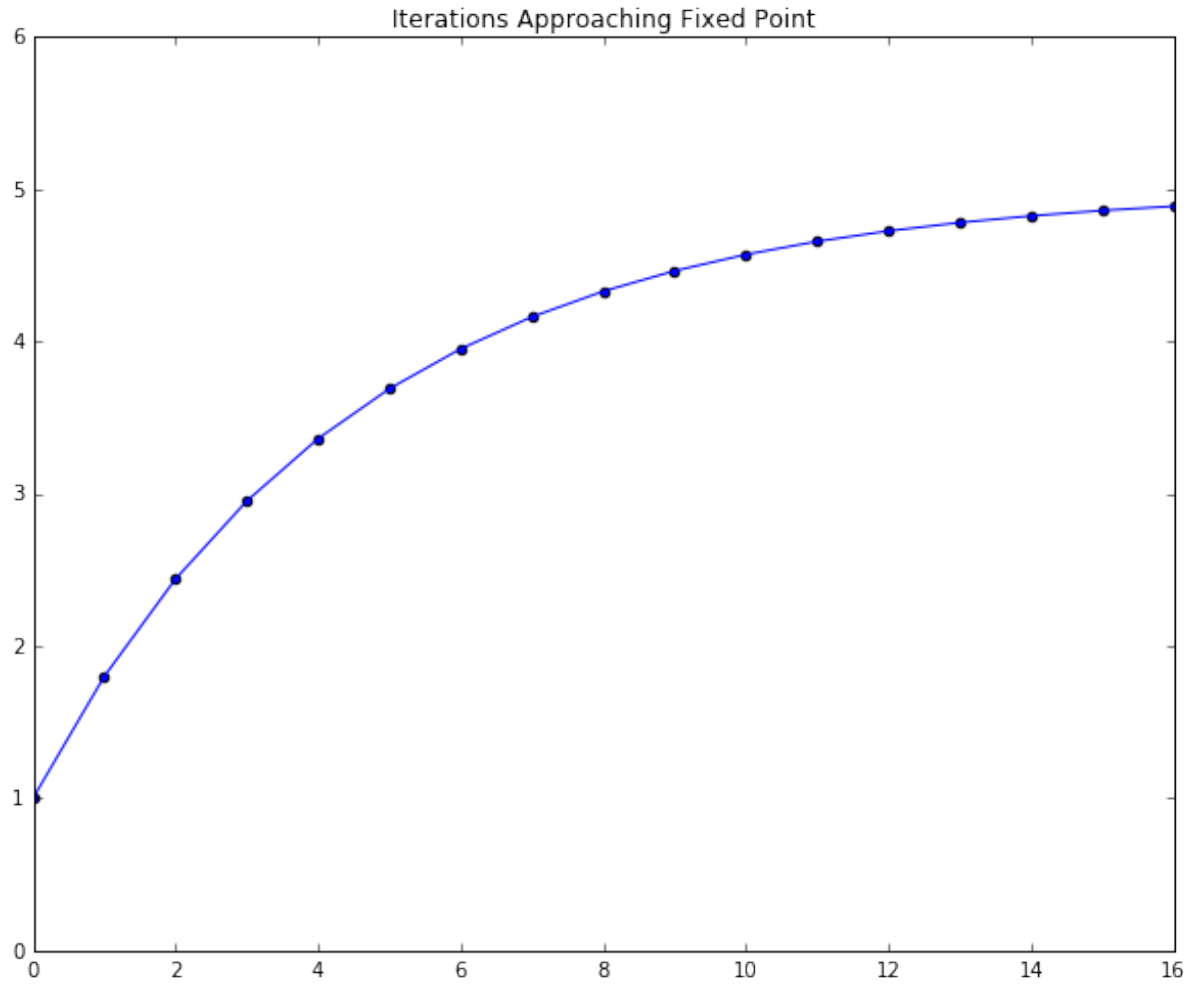
```
1.8
2.4400000000000004
2.9520000000000004
3.3616000000000006
3.6892800000000006
3.9514240000000007
4.161139200000001
4.328911360000001
4.4631290880000005
4.570503270400001
4.656402616320001
4.725122093056001
4.780097674444801
4.824078139555841
4.859262511644673
4.887410009315738
4.90992800745259
4.927942405962073
4.942353924769659
4.953883139815727
4.9631065118525814
4.9704852094820655
4.976388167585652
4.981110534068522
4.984888427254818
4.987910741803855
4.990328593443085
4.992262874754468
4.993810299803575
4.99504823984286
4.996038591874289
4.996830873499431
4.997464698799545
4.997971759039636
4.998377407231709
4.998701925785367
4.998961540628294
4.999169232502635
4.999335386002109
4.999468308801687
4.999574647041349
4.9996597176330795
4.999727774106464
4.999782219285171
4.999825775428137
```

```
4.99986062034251
4.999888496274008
4.999910797019206
4.9999286376153655
```

```
In [4]: def f(n):
            return 0.8*n+1
```

```
In [5]: n = 1; y = [n]
        x = range(0,20)
        for i in x[1:]:
            n = f(n)
            y.append(n)
        plt.figure(figsize=(10,8))
        plt.title('Iterations Approaching Fixed Point')
        plt.scatter(x,y)
        plt.plot(x,y)
        plt.xlim(0,16)
        plt.ylim(0,6)
```

```
Out[5]: (0, 6)
```



1.10 Fixed Point

Here, we would say 5 is our fixed point. This is born out by solving the equation:

$$x = 0.8x + 1$$

$$x = 5$$

Similarly, let's examine the behavior of the system created by iterating:

$$n_{i+1} = n_i - n_i^2 + 2$$

With a starting value of $n = 1.4$.

```
In [6]: n = 1.4
        for i in range(1,10):
            n = n - n*n + 2
            print(n)
```

```
1.4400000000000002
1.3663999999999998
1.4993510400000003
1.2512974988509178
1.685552068220355
0.8444662935384386
2.13134297261589
-0.41127989430324874
1.4195689542386598
```

Hmm... Something interesting is happening here. Let's look a little further down the line...

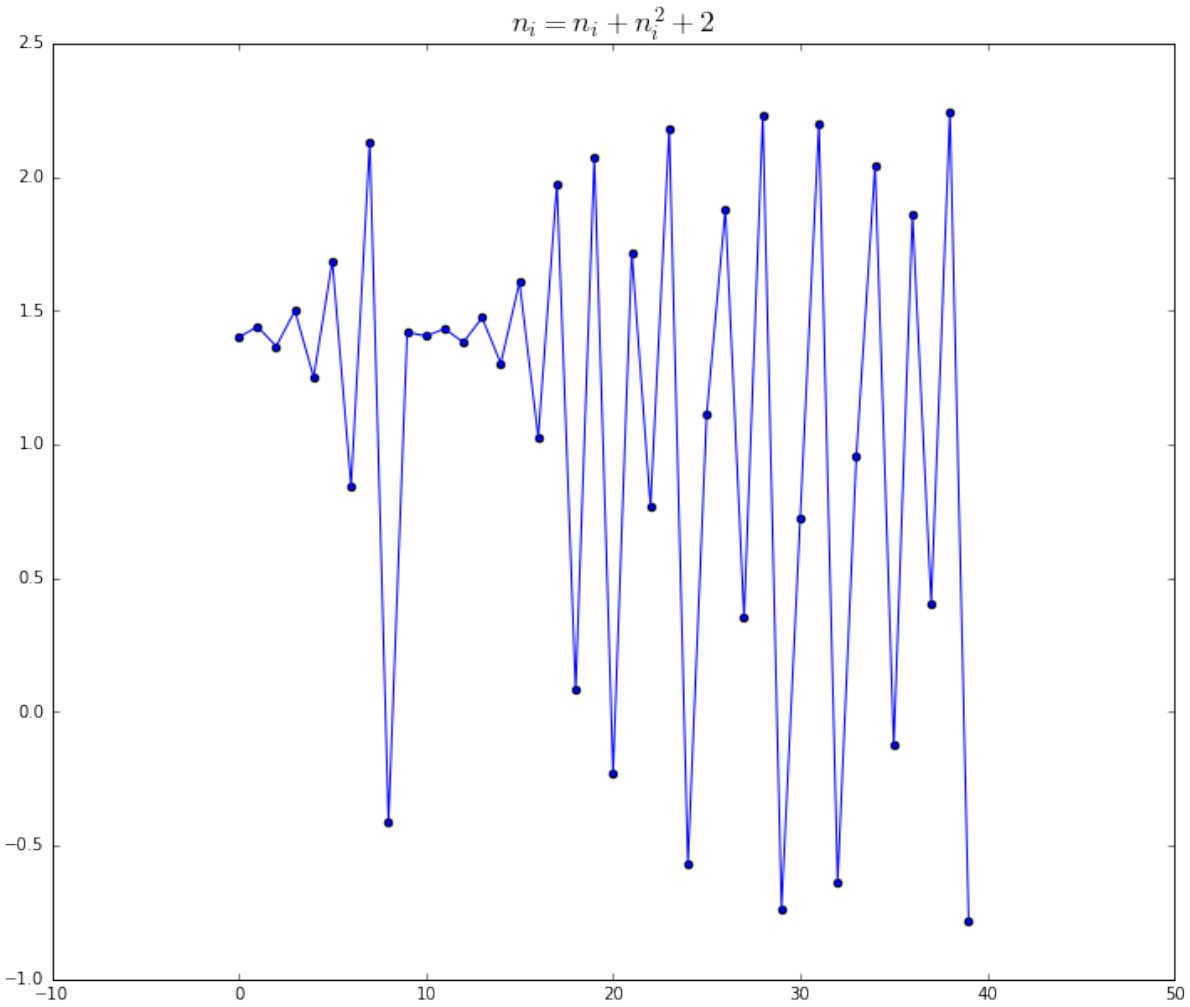
```
In [7]: n = 1.4
        for i in range(1,50):
            n = n - n*n + 2
            print(n)
```

```
1.4400000000000002
1.3663999999999998
1.4993510400000003
1.2512974988509178
1.685552068220355
0.8444662935384386
2.13134297261589
-0.41127989430324874
1.4195689542386598
1.4043929384004177
1.4320734129714583
1.3812391528317374
1.4734175555164017
1.3024582626124728
1.6060607367649717
1.026629646586928
1.9726612153357272
0.08126894484589875
2.074664303449533
-0.22956766855820243
1.717731016994549
0.7671311702494212
2.1786409378811746
-0.5678353983305899
1.1097275620721503
1.8782323000495522
0.3504757271001211
2.2276424918137625
-0.734748579520466
0.7253959453721914
2.199196667809776
-0.6372693158958462
0.956618503121794
```

```
2.0414995426068123
-0.12622083985701193
1.8578474597287786
0.40625027610810305
2.241210989270193
-0.7818157091552842
0.606948487762736
2.238562020965264
-0.7725979007428236
0.6304945830249584
2.2329711638011425
-0.7531890545662865
0.6795171935152571
2.2177735772324056
-0.7007460626378159
0.8082088930597822
```

In [8]:
```
def f(n):
    return n - n*n + 2

n=1.4
y=[n]
x = range(0,40)
for i in x[1:]:
    n = f(n)
    y.append(n)
plt.figure(figsize=(12,10))
plt.scatter(x,y)
plt.plot(x,y)
plt.title('$n_i = n_i + n_i^2 + 2$', fontsize=18)
```

Out[8]:
```
<matplotlib.text.Text at 0x10abc56d8>
```



The terms seem to bouncing around in an organized manner. If we solve for the fixed points, we see that $n = \pm\sqrt{2}$. Let's look at these examples from a different perspective with a cobweb diagram.

Cobweb Plot

A different way to visualize the behavior of the iterations is to look at what is called a cobweb diagram. The idea is to plot the line $y = x$ on the same axes as a continuous representation of the discrete system. Here, this results in graphing:

$$y1 = x \quad \text{and} \quad y2 = 0.8x + 1$$

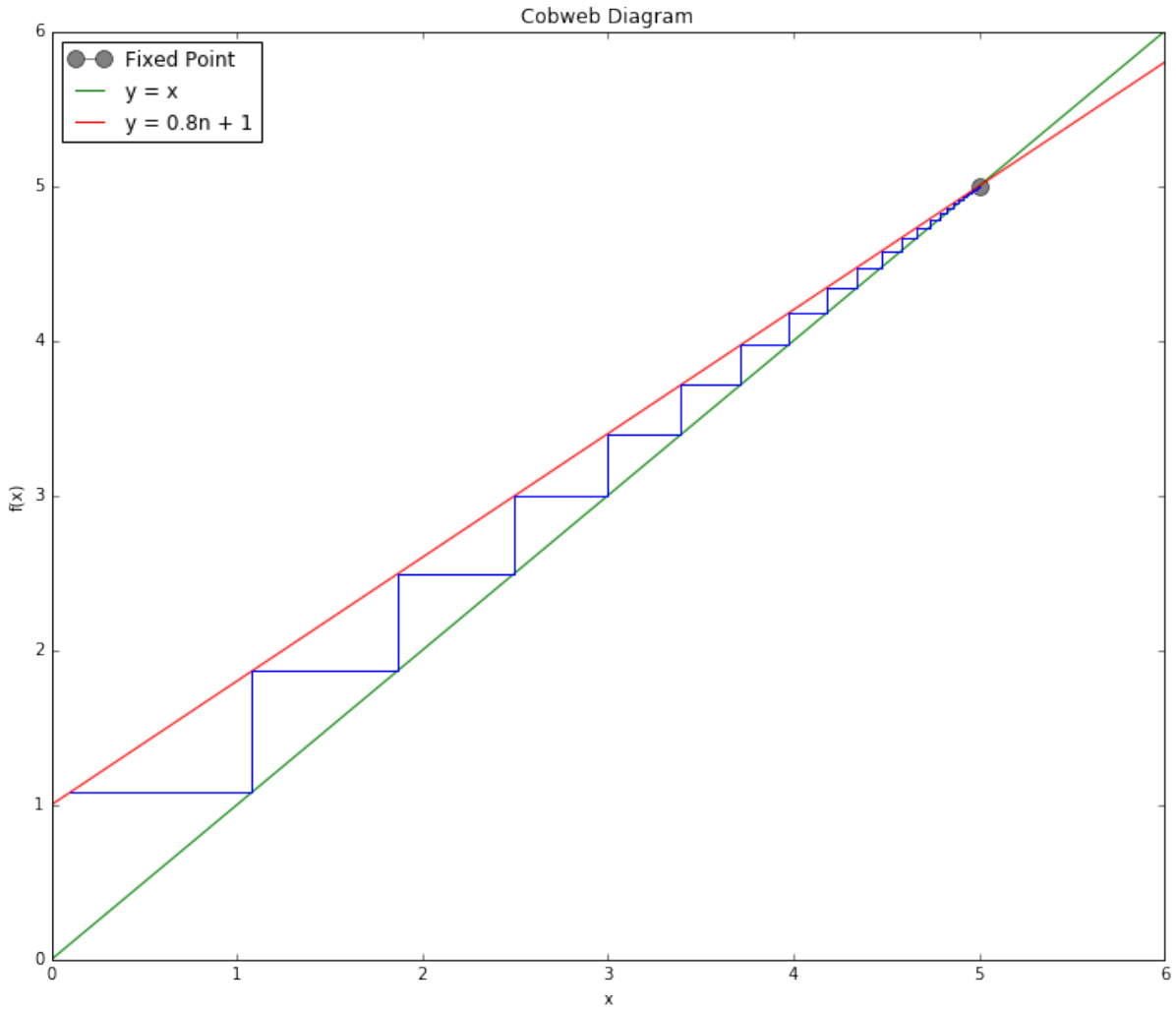
We then start at $x = 0$ on $y1$ draw a vertical line to $y2$, a horizontal line to $y1$, a vertical line to $y2$, repeat...

The results demonstrate not only where the fixed point is based on the intersection, but also give use insight into whether or not these points are repelling or attracting. For example, our fixed point of 5 is an attracting one, as it draws our pencil towards it no matter where we start our iterations.

```
In [9]: def f(n):
            return 0.8*n + 1

        # return f^n(x)
        def func_n(x,n):
            for i in range(0,n):
                x = f(x)
            return x
```

```
def plot_graphical(x0,n):
    xv = np.linspace(0.0,10.0,2*n)  # create array for points xvalue
    yv = np.linspace(0.0,10.0,2*n)  # create array for points yvalue
    x =x0
    for i in range(0,n):  #iterate
        xv[2*i] = x  # first point is (x,f(x))
        x = f(x)
        yv[2*i] = x
        xv[2*i+1] = x #second point is (f(x),f(x))
        yv[2*i+1] = x
    plt.plot(xv,yv,'b')  # connect up all these points blue


plt.figure(figsize=(12,10))
plt.xlabel('x')
plt.ylabel('f(x)')
plt.plot(5,5,  marker = 'o', markersize=10, c='grey',label = 'Fixed Point')
xcon = np.arange(0,7, 0.1)   # to plot function
plt.plot(xcon,xcon, 'g', label = 'y = x')#y=x plotted gree
alpha=0.81
ycon = f(xcon)                 # function computed
plt.plot(xcon,ycon, 'r', label='y = 0.8n + 1')             # function plotted red
plot_graphical(0.1,500)
plt.legend(loc='best')
plt.xlim(0,6)
plt.ylim(0,6)# cobweb plot, 0.3 is initial condition
plt.title('Cobweb Diagram')
```

Out[9]: <matplotlib.text.Text at 0x10a751278>



Different Behavior

We find more interesting examples from the earlier work with the sequences that demonstrated period 2 and period 3 behavior. Let's examine the cobweb diagram for

$$f_{n+1} = f_n - f_n^2 + 2$$

```
In [10]: def f(n):
             return n-n*n + 2

         # return f^n(x)
         def func_n(x,n):
             for i in range(0,n):
                 x = f(x)
             return x


         def plot_graphical(x0,n):
             xv = np.linspace(0.0,10.0,2*n)  # create array for points xvalue
             yv = np.linspace(0.0,10.0,2*n)  # create array for points yvalue
             x =x0
             for i in range(0,n):  #iterate
                 xv[2*i] = x  # first point is (x,f(x))
                 x = f(x)
                 yv[2*i] = x
```

```
        xv[2*i+1] = x #second point is (f(x),f(x))
        yv[2*i+1] = x
    plt.plot(xv,yv)  # connect up all these points blue


plt.figure(figsize=(12,10))
plt.xlabel('x')
plt.ylabel('f(x)')
xcon = np.arange(0,7, 0.1)   # to plot function
plt.plot(xcon,xcon, 'g')#y=x plotted gree
alpha=0.81
ycon = f(xcon)                  # function computed
plt.plot(xcon,ycon, 'r')             # function plotted red
plot_graphical(0.1,5000)
plt.xlim(-1,2.5)
plt.ylim(-1,2.5)# cobweb plot, 0.3 is initial condition
```

Out[10]: (-1, 2.5)